Convexification in global optimization

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Introduction: Global optimization

The general global optimization paradigm

General optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \subseteq \mathbb{R}^n, \\ & x \in [l, u], \end{array}$$

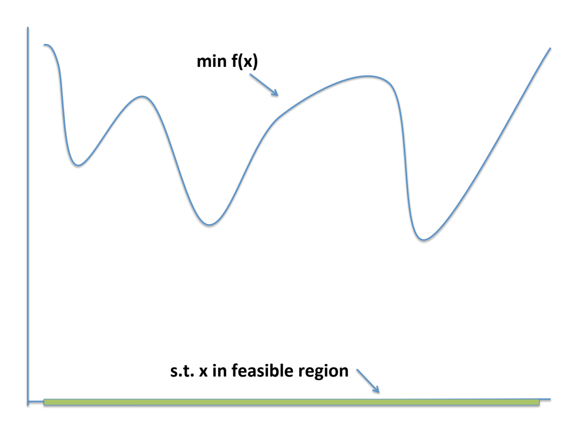
where

1. f is not necessarily a convex function, S is not necessarily a convex set.
2. Ideal goal: Find a globally optimal solution: x^*, i.e. $x^* \in S \cap [l, u]$ such that $OPT := f(x^*) \leq f(x)\ \forall x \in S \cap [l, u]$.
3. What we will usually settle for: $x^* \in S \cap [l, u]$ (may be approximately feasible) and a lower bound: LB such that:

$$x^* \in S \cap [l, u] \text{ and gap} := \frac{f(x^*) - LB}{LB} \text{ is "small" }.$$

Solving using Branch-and Bound

Branch-and-bound



Solving using Branch-and Bound

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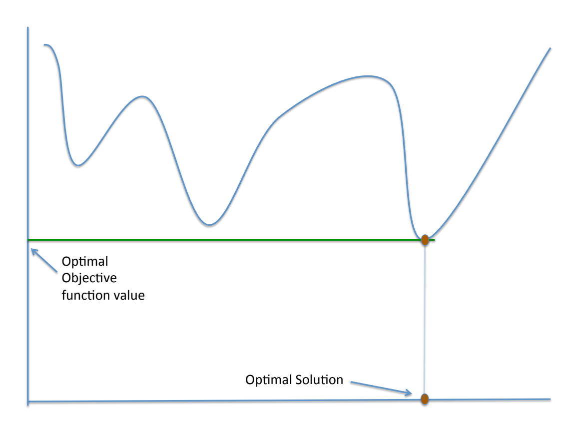

Solving using Branch-and Bound

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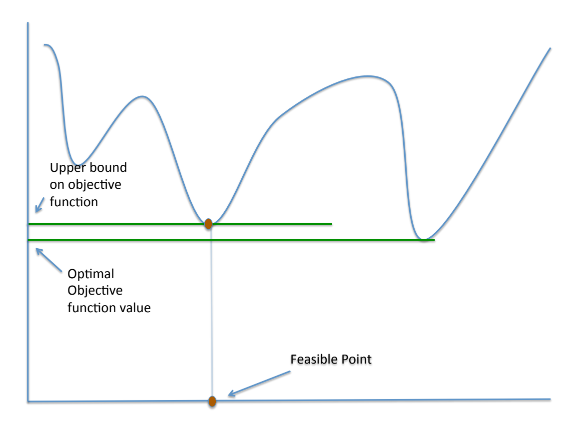
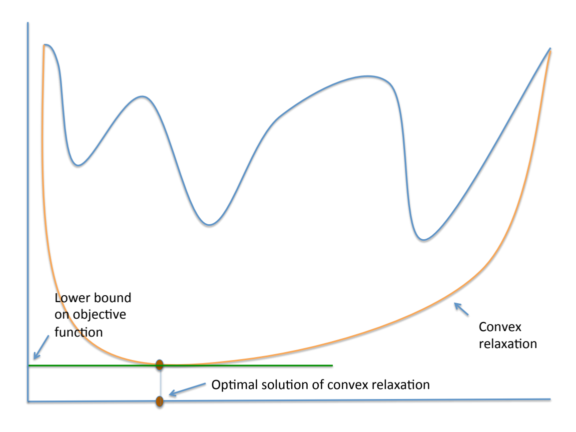


Solving using Branch-and Bound

Branch-and-bound



Solving using Branch-and Bound

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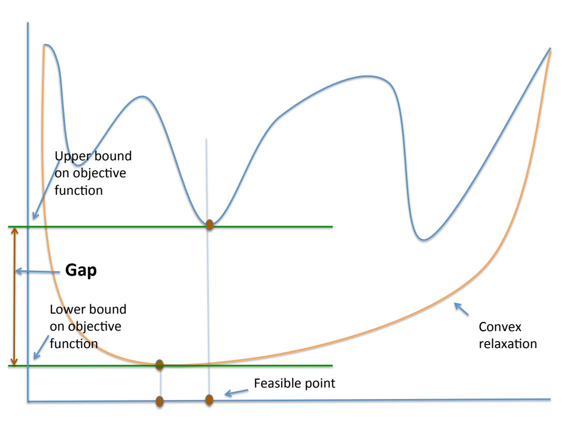

Solving using Branch-and Bound

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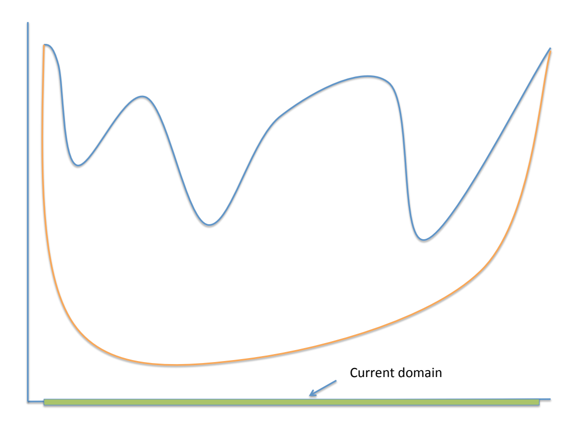
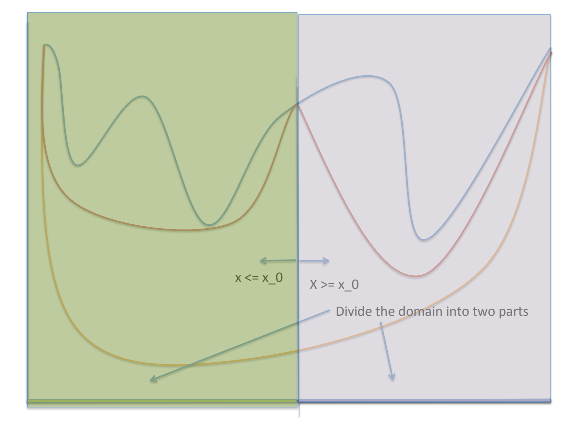


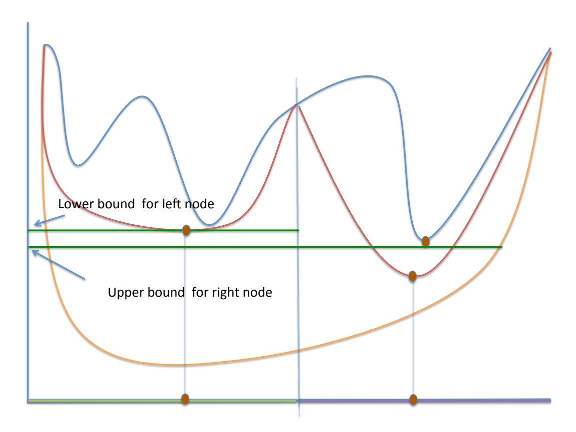
Solving using Branch-and Bound

Branch-and-bound



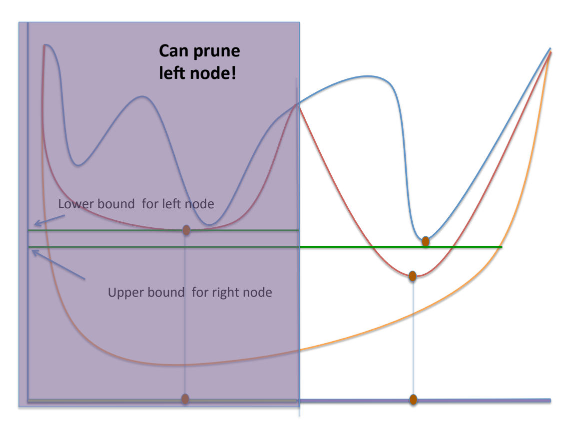
Solving using Branch-and Bound

Branch-and-bound



Solving using Branch-and Bound

Branch-and-bound



Discussion of Branch-and-bound algorithm

- The method works because: As the domain becomes "smaller" in the nodes, we are able to get a better (tighter) lower bound on $f(x)$. (♣)
- Usually S is not a convex set, then we need to obtain both: (1) a convex function that lower bounds $f(x)$ and (2) A convex relaxation of S.
- Our task is to obtain:
  (1) Machinery for obtaining"Good" lower bounding function that are convex and satisfying (♣)
  (2) "Good" convex relaxation of non-convex sets $S \cap [l, u]$.

Our goals for the next few hours

We want to study "convexification" for:

Quadrically constrainted quadratic program (QCQP)

$$\begin{aligned} \min \quad & x^\top Q x + c^\top x \\ \text{s.t.} \quad & x^\top Q^i x + (a^i)^\top x \le b_i \;\; \forall \;\; i \in [m] \\ & x \in [l, u], \end{aligned}$$

Very general model:

- Bounded polynomial optimization (replace higher order terms by quadratic terms by introducing new variables). For example:

$$xyz \le 3 \Leftrightarrow xy = w, wz \le 3.$$

- Bounded integer programs (including $0-1$ integer programs). For example:

$$x \in \{0, 1\} \Leftrightarrow x^2 - x = 0$$

Our goals for the next few hours

- Beautiful theory of **Lasserre hierarchy** which gives convex hulls via a hierarchy of Semi-definite programs (SDPs). (Also called the sums-of-square approach). We are not covering this theory. ☹
- Instead we will consider simple functions and simple sets that are relaxations of general QCQPs are consider their "convexification": You can think of this as the **MILP-approach**. Even though there are nice hierarchies for obtaining convex hulls in IP, in practice, we construct linear programming relaxations within branch-and-bound algorithm, which are often strengthened by addition of constraints obtained from the convexification of simple substructures.
- There will be other connections with integer programming...
- Usually, we will stick to **linear programming (LP)** or **second order cone representable (SOCr)** convex functions and sets for our convex relaxations.

Contribution of many people

- Warren Adams
- Claire S. Adjiman
- Shabbir Ahmed
- Kurt Anstreicher
- Gennadiy Averkov
- Harold P. Benson
- Daniel Bienstock
- Natashia Boland
- Pierre Bonami
- Samuel Burer
- Kwanghun Chung
- Yves Crama
- Danial Davarnia
- Alberto Del Pia
- Marco Duran
- Hongbo Dong
- Christodoulos A. Floudas
- Ignacio Grossmann
- Oktay Günlük
- Akshay Gupte
- Thomas Kalinowski
- Fatma Kılınç-Karzan
- Aida Khajavirad
- Burak Kocuk
- Jan Kronqvist
- Jon Lee
- Adam Letchford

Contribution of many people

- Jeff Linderoth
- Leo Liberti
- Jim Luedtke
- Marco Locatelli
- Andrea Lodi
- Alex Martin
- Clifford A. Meyer
- Garth P. McCormick
- Ruth Misener
- Gonzalo Munoz
- Mahdi Namazifar
- Jean-Philippe P. Richard
- Fabian Rigterink
- Anatoliy D. Rikun
- Nick Sahinidis
- Hanif Sherali
- Lars Schewe
- Felipe Serrano
- Suvrajeet Sen
- Emily Speakman
- Fabio Tardella
- Mohit Tawarmalani
- Hoáng Tuy
- Juan Pablo Vielma
- Alex Wang

And many more! I apologize in advance if I miss any citations. This is not intentional.

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Convex envelope: Definition and some properties

Definition: Convex envelope

Given $S \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$, we want:

- A function $g : \mathbb{R}^n \to \mathbb{R}$ that is an under estimator of f over S and,
- g should be convex.

Because (pointwise) supremum of a collection of convex functions is a convex function, we can achieve "the best possible convex under estimator" as follows:

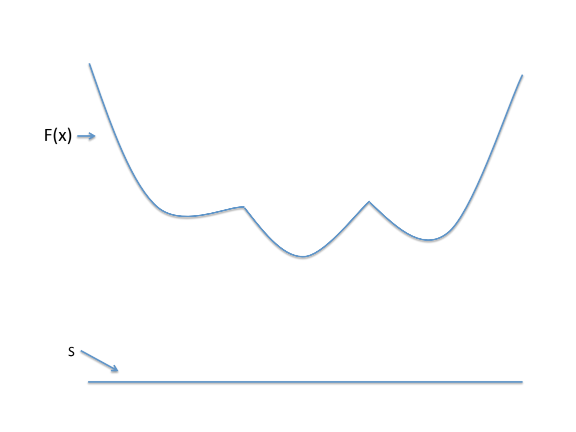
Definiton: Convex envelope

Given a set $S \subseteq \mathbb{R}^n$ and a function $f : S \to \mathbb{R}$, the convex envelope denoted as $\text{conv}_S(f)$ is:

$$\text{conv}_S(f)(x) = \sup\{g(x) \mid g \text{ is convex on } \text{conv}(S) \text{ and } g(y) \leq f(y) \; \forall y \in S\}.$$

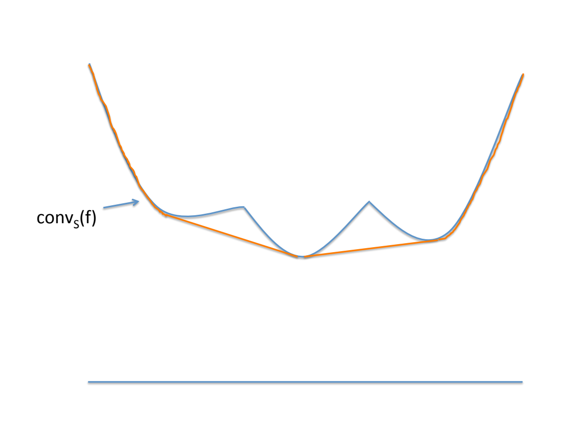
Convex envelope example

Convex envelope



Convex envelope example

Convex envelope



Another way to think about convex envelope

Definiton: Convex Envelope

Given a set $S \subseteq \mathbb{R}^n$ and a function $f : S \to \mathbb{R}$,

$\text{conv}_S(f)(x) = \sup\{g(x) \,|\, g \text{ is convex on conv}(S) \text{ and } g(y) \leq f(y)\ \forall y \in S\}.$

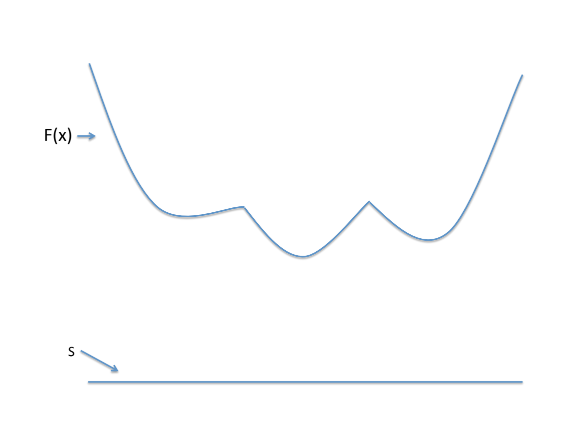
Proposition (1)

Given a set $S \subseteq \mathbb{R}^n$ and a function $f : S \to \mathbb{R}$, let $\text{epi}_S(f) := \{(w, x) \,|\, w \geq f(x), x \in S\}$ denote the epigraph of f restricted to S. Then the convex envelope is:

$$\text{conv}_S(f)(x) = \inf\{y \,|\, (y, x) \in \text{conv}(\text{epi}_S(f))\}. \quad (1)$$

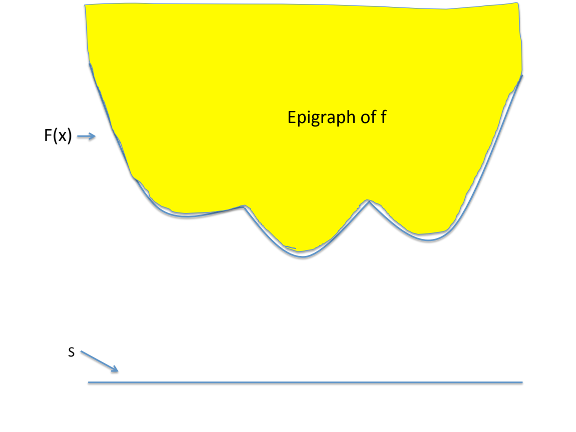
Convex envelope example contd.

Convex envelope



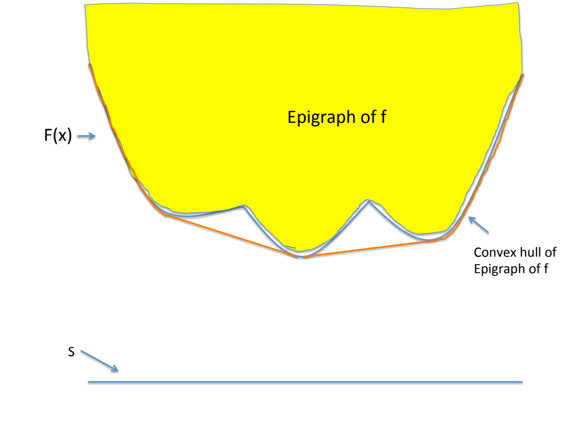
Convex envelope example contd.

Convex envelope



Convex envelope example contd.

Convex envelope



A simple property of convex envelope

Proposition (1)

$$\text{conv}_S(f)(x) = \inf\{y \mid (y, x) \in \text{conv}(\text{epi}_S(f))\}.$$

Corollary (1)

If x^0 is an extreme point of S, then $\text{conv}_S(f)(x^0) = f(x^0)$.

Proof.

We verify the contrapositive:

- Consider any $\hat{x} \in S$. If $\text{conv}_S(f)(\hat{x}) < f(\hat{x})$, then (via Proposition (1)) there must be $\{x^i\}_{i=1}^{n+2} \in S$:

$$\hat{x} = \sum_{i=1}^{n+2} \lambda_i x^i, \quad f(\hat{x}) > \sum_{i=1}^{n+2} \lambda_i f(x^i),$$

  where $\lambda \in \Delta$ (i.e. $\lambda_i \geq 0 \ \forall i \in [n+2]$, $\sum_{i=1}^{n+2} \lambda_i = 1$).
- If $\hat{x} = x^i \ \forall\ i$, then $f(\hat{x}) \not> \sum_{i=1}^{n+2} \lambda_i f(x^i) \Rightarrow x \neq x^i \Rightarrow \hat{x}$ is not extreme.

When does extreme points of S describe the convex envelope of $f(x)$?

Let S be a polytope.

- We know now that $\text{conv}_S(f)(x^0) = f(x^0)$ for extreme points.
- For $x^0 \in S$ and $x^0 \notin \text{ext}(S)$, we know that

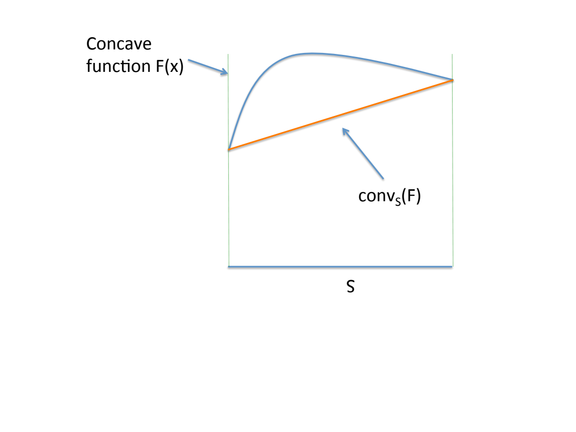
$$\text{conv}_S(f)(x^0) = \inf\left\{y \,|\, y = \sum_i \lambda_i f(x^i), x^0 = \sum_i \lambda_i x^i, x^i \in S, \lambda \in \Delta\right\}.$$

- It would be nice (why?) if:

$$\text{conv}_S(f)(x^0) = \inf\left\{y \,|\, y = \sum_i \lambda_i f(x^i), x^0 = \sum_i \lambda_i x^i, x^i \in \text{ext}(S), \lambda \in \Delta\right\}.$$

Concave function work: proof by example

Concave function



Sufficient condition for polyhedral convex envelope of $f(x)$: When f is edge concave

Definiton: Edge concave function

Given a polytope $S \subseteq \mathbb{R}^n$. Let $S_D = \{d_1, \dots, d_k\}$ be a set of vectors such that for each edge E (one-dimensional face) of S, S_D contains a vector parallel to E. Let $f : S \to \mathbb{R}^n$ be a function. We say f is edge concave for S if it is concave on all line segments in S that are parallel to an edge of S, i.e., on all the sets of the form:

$$\{y \in S \,|\, y = x + \lambda d\},$$

for some $x \in S$ and $d \in S_D$.

Example of edge concave function

Bilnear function

- $S := \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x, y \leq 1\}$.
- $S_d = \{(0, 1), (1, 0)\}$.
- $f(x, y) = xy$ is linear for all segments in S that are parallel to an edge of S.
- Therefore f is a edge concave function over S.

Note: $f(x, y) = xy$ is not concave.

Polyhedral convex envelope of $f(x)$: f is edge concave

Theorem (Edge concavity gives polyhedral envelope [Tardella (1989)])

Let S be a polytope and $f : S \to \mathbb{R}^n$ is an edge concave function. Then $\text{conv}_S(f)(x) = \text{conv}_{ext(S)}(f)(x)$, *where*

$$\text{conv}_{ext(S)}(f)(x) := \min\left\{ y \,|\, y = \sum_i \lambda_i f(x^i), x = \sum_i \lambda_i x^i, x^i \in \text{ext}(S), \lambda \in \Delta \right\}.$$

Corollary [Rikun (1997)]

Let $f = \prod_i x_i$ and $S = [l, u]$. Then $\text{conv}_S(f)(x) = \text{conv}_{ext(S)}(f)(x)$.

Polyhedral convex envelope of $f(x)$: f is edge concave

Theorem (Edge concavity gives polyhedral envelope [Tardella (1989)])

Let S be a polytope and $f : S \to \mathbb{R}^n$ is an edge concave function. Then $\text{conv}_S(f)(x) = \text{conv}_{ext(S)}(f)(x)$, *where*

$$\text{conv}_{ext(S)}(f)(x) := \min\left\{ y \,|\, y = \sum_i \lambda_i f(x^i), x = \sum_i \lambda_i x^i, x^i \in \text{ext}(S), \lambda \in \Delta \right\}.$$

Proof sketch

- Claim 1: Since f is edge concave, we obtain: $f(x) \geq \text{conv}_{ext(S)}(f)(x)$ for all $x \in S$.
- Claim 2: If $f(x) \geq \text{conv}_{ext(S)}(f)(x)$, then

$$\text{conv}_S(f)(x) = \text{conv}_{ext(S)}(f)(x).$$

Proof of Claim 1

To prove: $f(x) \geq \text{conv}_{ext(S)}(f)(x)$

Let $\hat{x} \in \text{rel.int}(F)$, F is a face of S. Proof by induction on the dimension of F.

- Base case: Consider $\hat{x}$ which belongs to a one-dimensional face of S, i.e. $\hat{x}$ belongs to an edge of f. Then since edge-concavity, we obtain that $f(\hat{x}) \geq \text{conv}_{ext(S)}(f)(\hat{x})$.
- Inductive step: Let F be a face of S where $\dim(F) \geq 2$. Consider $\hat{x} \in \text{rel.int}(F)$. If we show that there is x^1, x^2 belonging to proper faces of F, such that $\hat{x} = \lambda_1 x^1 + \lambda_2 x^2$, $\lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geq 0$, and $f(\hat{x}) \geq \lambda_1 f(x^1) + \lambda_2 f(x^2)$. Then applying this argument recursively to $f(x^1)$ and $f(x^2)$ we obtain the result.
- Indeed, consider an edge of F and let d be the direction of this edge. Then there exists $\mu_1, \mu_2 > 0$ such that: $\hat{x} + \mu_1 d$ and $\hat{x} - \mu_2 d$ belong to lower dimensional faces of F. Now on this segment edge-concavity = concavity, so we are done.

Proof of Claim 2

$$\text{conv}_S(f)(x^0) = \inf\left\{y \,|\, y = \sum_i \lambda_i f(x^i), x^0 = \sum_i \lambda_i x^i, x^i \in S, \lambda \in \Delta\right\}.$$

$$\text{conv}_{ext(S)}(f)(x^0) = \inf\left\{y \,|\, y = \sum_i \lambda_i f(x^i), x^0 = \sum_i \lambda_i x^i, x^i \in \text{ext}(S), \lambda \in \Delta\right\}.$$

To prove: $f(x) \geq \text{conv}_{ext(S)}(f)(x)$, implies $\text{conv}_S(f)(x) = \text{conv}_{ext(S)}(f)(x)$

- Note that $\text{conv}_S(f) \leq \text{conv}_{\text{ext}(S)}(f)$ (by definition), so it is sufficient to prove $\text{conv}_S(f) \geq \text{conv}_{\text{ext}(S)}(f)$.
- Indeed, observe that

$$\begin{aligned} \text{conv}_S(f) &\geq \text{conv}_S\left(\text{conv}_{\text{ext}(S)}(f)\right) \\ &= \text{conv}_{\text{ext}(S)}(f) \end{aligned}$$

  where the first inequality because of Claim 1, $f(x) \geq \text{conv}_{ext(S)}(f)(x)$, and the second inequality because $\text{conv}_{\text{ext}(S)}(f)$ is a convex function.

3
Convex hull of simple sets

3.1
McCormick envelope

McCormick envelope

$$P := \{(w,x,y) \,|\, w = xy, 0 \le x, y \le 1\}$$

We want to find conv(P).

- $P = \{(w,x,y) \,|\, \underbrace{w = xy}_{f(x,y)=xy}, \underbrace{0 \le x, y \le 1}_{S}\}$
- So we need to find the convex envelope (and similarly, concave envelope) of $f(x,y) = xy$ over $x, y \in [0,1]$).
- By previous section **result on edge-concavity**, we only need to consider the extreme points of $S = [0,1]^2$.
- $\text{conv}(P) = \text{conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$

$$\text{conv}(P) = \{(w,x,y) \,|\, \underbrace{w \ge 0, w \ge x + y - 1, w \le x, w \le y}_{\text{McCormick Envelope}}\}.$$

Alternative proof of validity of McCormick envelope

- $\underbrace{(x-0)(y-0)}_{\text{product of 2 non-negative trms}} \geq 0 \Leftrightarrow xy \geq 0 \underbrace{\Rightarrow}_{\text{replace } w=xy} w \geq 0.$
- $\underbrace{(1-x)(1-y)}_{\text{product of 2 non-negative trms}} \geq 0 \Leftrightarrow xy \geq x+y-1 \Rightarrow w \geq x+y-1.$
- $(x-0)(1-y) \geq 0 \Rightarrow w \leq x.$
- $(1-x)(y-0) \geq 0 \Rightarrow w \leq y.$
- This is the Reformulation-linearization-techique (RLT) view point (Sherali-Adams).

Our first convex relaxation of QCQP

$$\begin{aligned}
(\text{QCQP}) : \min \;\; & x^T A_0 x + a_0^T x \\
\text{s.t.} \;\; & x^T A_k x + a_k^T x \leq b_k \quad k = 1, \dots, K \\
& l \leq x \leq u
\end{aligned}$$

$$\begin{aligned}
(\text{Lifted QCQP}) : \min \;\; & \underbrace{A_0 \cdot X}_{\sum_{i,j} (A_0)_{ij} X_{ij}} + a_0^T x \\
\text{s.t.} \;\; & \underbrace{A_k \cdot X}_{\sum_{i,j} (A_k)_{ij} X_{ij}} + a_k^T x \leq b_k \quad k = 1, \dots, K \\
& l \leq x \leq u \\
& \boxed{X = x x^\top} <--- \text{Nonconvexity}
\end{aligned}$$

(Note: X is the “outer product” of x, i.e. X is $n \times n$)

Our first convex (LP) relaxation of QCQP

$$
\begin{aligned}
\text{(QCQP)} : \min \quad & x^T A_0 x + a_0^T x \\
\text{s.t.} \quad & x^T A_k x + a_k^T x \le b_k \quad k = 1, \dots, K \\
& l \le x \le u
\end{aligned}
$$

$$
\begin{aligned}
\text{(Lifted QCQP)} : \min \quad & A_0 \cdot X + a_0^T x \\
\text{s.t.} \quad & A_k \cdot X + a_k^T x \le b_k \quad k = 1, \dots, K \\
& l \le x \le u \\
& \boxed{X = xx^T}
\end{aligned}
$$

McCormick (LP) Relaxation: replace $X = xx^\top$ above by:

$$
\begin{aligned}
X_{ij} &\ge l_i x_j + l_j x_i - l_i l_j \\
X_{ij} &\ge u_i x_j + u_j x_i - u_i u_j \\
X_{ij} &\le l_i x_j + u_j x_i - l_i u_j \\
X_{ij} &\le u_i x_j + l_j x_i - u_i l_j
\end{aligned}
$$

Semi-definite programming (SDP) relaxation of QCQPs

$$\begin{aligned}(\text{QCQP}) : \min\ & x^T A_0 x + a_0^T x \\ \text{s.t.}\ & x^T A_k x + a_k^T x \leq b_k \quad k = 1, \dots, K \\ & l \leq x \leq u\end{aligned}$$

$$\begin{aligned}(\text{Lifted QCQP}) : \min\ & A_0 \cdot X + a_0^T x \\ \text{s.t.}\ & A_k \cdot X + a_k^T x \leq b_k \quad k = 1, \dots, K \\ & l \leq x \leq u \\ & \boxed{X = xx^T}\end{aligned}$$

SDP Relaxation: replace $X - xx^\top = 0$ above by:

$$X - xx^\top \in \text{cone of positive-semi definite matrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix} \in \text{cone of positive-semi definite matrix.}$$

Comments

- The SDP relaxation is the first level of the sum-of-square hierarchy. (We will not discuss this more here)
- The McCormick relaxation is first (basic) level of the RLT hireranchy.
- The McCormick relaxation and the SDP relaxation are incomparable. So many times if one is able to solve SDPs, both the relaxations are thrown in together.
- Note that the McCormick relaxation has the (♣) property, i.e. as the bounds $[l, u]$ get tighter, the McCormick envelopes gets better. In particular, if $l = u$, then the McComick envelope is exact. Therefore, we can obtain "asymptotic convergence of lower and upper bound" using a branch and bound tree with McCormick relaxation, as the size of the tree goes off to infinity.

3.2
Extending the McCormick envelope ideas

Extending the McCormick envelope argument: Using extreme points of S to construct convex hull

$$\begin{aligned}(\text{Lifted QCQP}): \min\ & A_0 \cdot X + a_0^T x \\ \text{s.t. }\ & A_k \cdot X + a_k^T x \leq b_k \quad k = 1, \dots, K \\ & 0 \leq x \leq 1 \\ & \boxed{X = xx^T}\end{aligned}$$

For now ignore the x_i^2 terms and consider the set:

$$Q := \left\{(X, x) \in \mathbb{R}^{\frac{n(n-1)}{2}} \times \mathbb{R}^n \,|\, X_{ij} = x_i x_j \forall i, j \in [n], i \neq j, x \in [0,1]^n\right\}$$

(Here $l = 0$ and $u = 1$ without loss of generality, by rescaling the variables.)

Extending the McCormick envelope argument: Using extreme points of S to construct convex hull

Theorem ([Burer, Letchford (2009)])

Consider the set

$$Q := \{(X, x) \in \mathbb{R}^{\frac{n(n-1)}{2}} \times \mathbb{R}^n \,|\, X_{ij} = x_i x_j \,\forall i, j \in [n], i \neq j, x \in [0,1]^n\}.$$

Then,

$$\text{conv}(Q) := \text{conv}\left(\left\{\underbrace{(X, x) \in \mathbb{R}^{\frac{n(n-1)}{2}} \times \mathbb{R}^n \,|\, X_{ij} = x_i x_j \,\forall i, j \in [n], i \neq j, x \in \{0,1\}^n}_{\text{Boolean quadric polytope}}\right\}\right).$$

Krein - Milman theorem

Theorem (Krein - Milman Theorem)

Let $S \subseteq \mathbb{R}^n$ be a compact set. Then $\mathrm{conv}(S) = \mathrm{conv}(\mathrm{ext}(S))$.

Proof of Theorem

Proof using "Extreme point of S argument"

- By Krein - Milman Theorem, It is sufficient to prove that the extreme points of Q:

$$Q := \{(X, x) \in \mathbb{R}^{\frac{n(n-1)}{2}} \times \mathbb{R}^n \,|\, X_{ij} = x_i x_j \, \forall i, j \in [n], i \neq j, x \in [0,1]^n\}$$

satisfy $x \in \{0,1\}^n$.

- Suppose $(\hat{X}, \hat{x}) \in Q$ is an extreme point of S. Assume by contradition $\hat{x}_i \notin \{0,1\}$. Consider the following points:

$$x_j^{(1)} = \begin{cases} \hat{x}_j & j \neq i \\ \hat{x}_i + \epsilon & j = i \end{cases} \qquad x_j^{(2)} = \begin{cases} \hat{x}_j & j \neq i \\ \hat{x}_i - \epsilon & j = i \end{cases}$$

$$X_{uv}^{(1)} = \begin{cases} \hat{X}_{uv} & u, v \neq i \\ \hat{x}_u x_v^{(1)} & v = i \end{cases} \qquad X_{uv}^{(2)} = \begin{cases} \hat{X}_{uv} & u, v \neq i \\ \hat{x}_u x_v^{(2)} & v = i \end{cases}$$

- Since there is no "square term", $X^{(\cdot)}$ perturbs linearly with perturbation of one component of $x^{(\cdot)}$.
- So $(\hat{X}, \hat{x}) = 0.5 \cdot (X^{(1)}, x^{(1)}) + 0.5 \cdot (X^{(2)}, x^{(2)})$, which is the required contradiction.

Consequence: Can use IP technology to obtain better convexification of QCQP!

$$
\begin{aligned}
\text{(Lifted QCQP)}: \min\ & A_0 \cdot X + a_0^T x \\
\text{s.t. }\ & A_k \cdot X + a_k^T x \le b_k \quad k = 1, \dots, K \\
& 0 \le x \le 1 \\
& \boxed{X = xx^T}
\end{aligned}
$$

Apart from the McCormick inequalities we can also add:

- Triangle inequality: $x_i + x_j + x_k - X_{ij} - X_{jk} - X_{ik} \le 1$ [Padberg (1989)]
- $\{0, \frac{1}{2}\}$ Chvatal-Gomory cuts for BQP recently used successfully by [Bonami, Günlük, Linderoth (2018)]

$BQP := \{(X, x) \,|\, X_{ij} \ge 0, X_{ij} \ge x_i + x_j - 1, X_{ij} \le x_i, X_{ij} \le j \;\forall\; (i,j) \in [n], x \in \{0,1\}^n\}$

4

Incorporating “data” in our sets

Introduction

$$\text{(Lifted QCQP)} : \min \quad A_0 \cdot X + a_0^T x$$
$$\text{s.t.} \quad A_k \cdot X + a_k^T x \leq b_k \quad k = 1, \ldots, K$$
$$0 \leq x \leq 1$$
$$\boxed{X = xx^T}$$

- We have explored convex hull of set of the form:

$$Q := \{(X, x) \in \mathbb{R}^{\frac{n(n-1)}{2}} \times \mathbb{R}^n \,|\, X_{ij} = x_i x_j \,\forall i, j \in [n], i \neq j, x \in [0, 1]^n\}$$

- Now we want to consider sets wich includes the data, for example: A_k's.

4.1
A packing-type bilinear knapsack set

A packing-type bilinear knapsack set

Consider the following set:

$$P := \{(x,y) \in [0,1]^n \times [0,1]^n \mid \sum_{i=1}^{n} a_i x_i y_i \leq b\},$$

where $a_i \geq 0$ for all $i \in [n]$.

The convex-hull of packing-type bilinear set

Proposition (3 Coppersmith, Günlük, Lee, Leung (1999))

Let $P := \{(x,y) \in [0,1]^n \times [0,1]^n \mid \sum_i a_i x_i y_i \leq b\}$. *Then*

$$\text{conv}(P) := \left\{ (x,y) \;\middle|\; \begin{array}{l} \underbrace{\begin{array}{l} \exists w, \sum_{i=1}^{n} a_i w_i \leq b, \\ w_i, x_i, y_i \in [0,1], w_i \geq x_i + y_i - 1, \end{array}}_{\text{Relaxed McCormick envelope}} \;\forall i \in [n] \end{array} \right\}.$$

- Convex hull is a polytope.
- Shows the power of McCormick envelopes.

Proof of Proposition(3): $\subseteq$

$$\mathrm{conv}(P) := \mathrm{Proj}_{x,y}\Bigg(\underbrace{\left\{(x,y,w)\;\middle|\; \begin{array}{l} \sum_{i=1}^{n} a_i w_i \leq b, \\ w_i, x_i, y_i \in [0,1], w_i \geq x_i + y_i - 1 \;\; \forall i \in [n] \end{array}\right\}}_{\mathrm{R}}\Bigg).$$

- Observe $P \subseteq \mathrm{Proj}_{x,y}(R) \Rightarrow \mathrm{conv}(P) \subseteq \mathrm{Proj}_{x,y}(R)$.

Proof of Proposition(3): $\text{conv}(P) \supseteq \text{Proj}_{x,y}(R)$

$$\text{conv}(P) := \text{Proj}_{x,y}\left(\underbrace{\left\{(x,y,w) \;\middle|\; \begin{array}{l} \sum_{i=1}^{n} a_i w_i \leq b, \\ w_i, x_i, y_i \in [0,1], w_i \geq x_i + y_i - 1 \;\; \forall i \in [n] \end{array}\right\}}_{\text{R}}\right).$$

It is sufficient to prove that the (x, y) component of extreme points of R belong to P.

Let $(\hat{w}, \hat{x}, \hat{y})$ be extreme point of R. For each i:

- If $\hat{w}_i = 0$, then $(\hat{x}_i, \hat{y}_i) \in \{(0,0), (0,1), (1,0)\}$, i.e. $\hat{x}_i\hat{y}_i = \hat{w}_i$.
- If $0 < \hat{w}_i < 1$, then $(\hat{x}_i, \hat{y}_i) \in \{(0,0), (0,1), (1,0), (1, \hat{w}_i), (\hat{w}_i, 1)\}$, i.e. $\hat{x}_i\hat{y}_i \leq \hat{w}_i$.
- If $\hat{w} = 1$, then $(\hat{x}_i, \hat{y}_i) \in \{(0,0), (1,0), (0,1), (1,1)\}$, i.e. $\hat{x}_i\hat{y}_i \leq \hat{w}_i$.

Thus, $\sum_{i=1}^{n} a_i \hat{x}_i \hat{y}_i \leq b$. $(\because a_i \geq 0 \;\; \forall i \in [n]\,)$

Proof of Proposition(3): $\text{conv}(P) \supseteq \text{Proj}_{x,y}(R)$

$$\text{conv}(P) := \text{Proj}_{x,y}\left(\underbrace{\left\{(x,y,w) \;\middle|\; \begin{array}{l} \sum_{i=1}^n a_i w_i \leq b, \\ w_i, x_i, y_i \in [0,1], w_i \geq x_i + y_i - 1 \;\; \forall i \in [n] \end{array}\right\}}_{\text{R}}\right).$$

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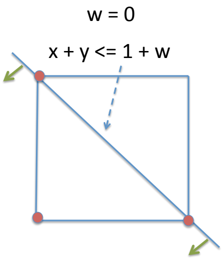
Thus, $\sum_{i=1}^n a_i\hat{x}_i\hat{y}_i \le b$. $(\because a_i \ge 0 \;\forall i \in [n]\,)$



Proof of Proposition(3): $\text{conv}(P) \supseteq \text{Proj}_{x,y}(R)$

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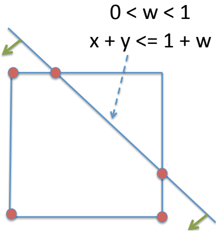

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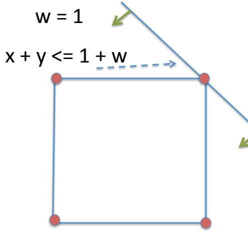
Thus, $\sum_{i=1}^n a_i\hat{x}_i\hat{y}_i \leq b$. ($\because a_i \geq 0\ \forall i \in [n]$)



4.2
Product of a simplex and a polytope

A commonly occuring set

$$S := \{(q, y, v) \in \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_1 n_2} \,|\, v_{ij} = q_i y_j \,\forall i \in [n_1], j \in [n_2], \underbrace{Ay \le b}_{y \in P}, \underbrace{q \in \Delta}_{\sum_{i=1}^{n_1} q_i = 1} \}.$$

Some applications:

- Pooling problem ([Tawarmalani and Sahinidis (2002)])
- General substructure in "discretize NLPs" ([Gupte, Ahmed, Cheon, D. (2013)])
- Network interdiction ([Davarnia, Richard, Tawarmalani (2017)])

Convex hull of S

Theorem (Sherali, Alameddine [1992], Tawarmalani (2010), Kılınç-Karzan (2011))

Let

$$S := \left\{ (q, y, v) \in \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_1 n_2} \;\middle|\; \begin{array}{l} v_{ij} = q_i y_j \forall i \in [n_1], j \in [n_2], \\ Ay \le b, \\ q \in \Delta \end{array} \right\}.$$

Then $\text{conv}(S) := \text{conv}\left(\cup_{i=1}^{n_1} \underbrace{\{(q, y, v) \,|\, q_i = 1, v_{ij} = y_j, y \in P\}}_{S_i} \right).$

Proof of Theorem: $\supseteq$

Theorem

Let

$$S := \left\{ (q, y, v) \in \mathbb{R}^{n_1}_{+} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_1 n_2} \;\middle|\; \begin{array}{l} v_{ij} = q_i y_j \, \forall i \in [n_1], j \in [n_2], \\ Ay \leq b, \\ q \in \Delta \end{array} \right\}.$$

Then $\text{conv}(S) := \text{conv}\left(\cup_{i=1}^{n_1} \underbrace{\{(q, y, v) \,|\, q_i = 1, v_{ij} = y_j, y \in P\}}_{S_i} \right)$.

Proof of $\supseteq$

- $S_i \subseteq S.\ \ \forall i \in [n_1]$
- $\cup_{i=1}^{n_1} S_i \subseteq S$.
- $\text{conv}(\cup_{i=1}^{n_1} S_i) \subseteq \text{conv}(S)$.

Proof of Theorem: ⊆

$$S := \{(q,y,v) \in \mathbb{R}^{n_1}_{+} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_1 n_2} \,|\, v_{ij} = q_i y_j \,\forall i \in [n_1], j \in [n_2], Ay \leq b, q \in \Delta\}$$

$$\text{conv}(S) := \text{conv}\left(\bigcup_{i=1}^{n_1} \underbrace{\{(q,y,v)\,|\, q_i = 1, v_{ij} = y_j, y \in P\}}_{S_i}\right).$$

Proof of ⊆

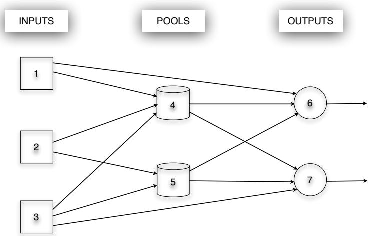
- Pick $(\hat{q},\hat{y},\hat{v}) \in S$. We need to show $(\hat{q},\hat{y},\hat{v}) \in \text{conv}(\bigcup_{i=1}^{n_1} S_i)$
- Let $I \subseteq [n_1]$ such that $\hat{q}_i \neq 0$ for $i \in I$. Then it is easy to verify, $(\hat{q},\hat{y},\hat{v})$ is the convex combination of the points of the form for $i_0 \in I$:

$$\left.\begin{array}{lcl} \tilde{q}^{i_0} & = & e_{i_0} \\ \tilde{y}^{i_0} & = & \hat{y} \\ \tilde{v}^{i_0}_{ij} & = & \begin{cases} \hat{y}_j & \text{if } i = i_0 \\ 0 & \text{if } i \neq i_0 \end{cases} \end{array}\right\} \in S_{i_0} \;\; \forall i_0 \in I$$

- $\Rightarrow (\hat{q},\hat{y},\hat{v}) \in \text{conv}(\bigcup_{i=1}^{n_1} S_i)$

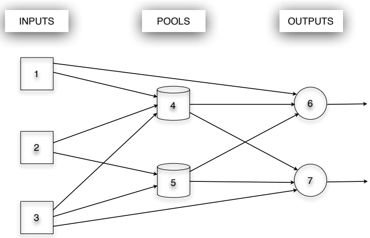
4.2.1
Application: Pooling problem

The Pooling Problem: Network Flow on Tripartite Graph



- Network flow problem on a tripartite directed graph, with three type of node: *Input* Nodes (I), *Pool* Nodes (L), *Output* Nodes (J).
- Send flow from input nodes via pool nodes to output nodes.
- Each of the arcs and nodes have capacities of flow.

The Pooling Problem: Network Flow on Tripartite Graph



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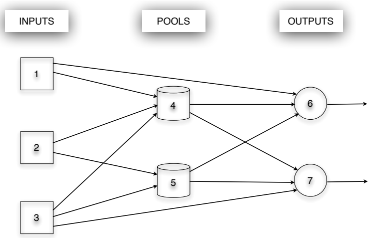

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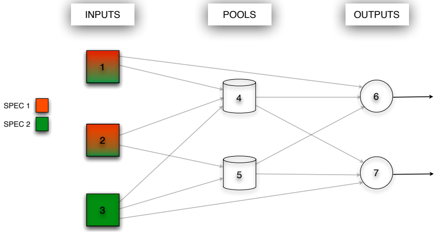
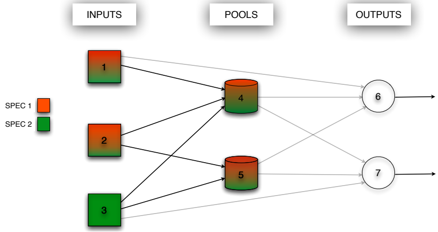
The Pooling Problem: Other Constraints



- Raw material has specifications (like sulphur, carbon, etc.).
- Raw material gets mixed at the pool producing new specification level at pools.
- The material gets further mixed at the output nodes.
- The output node has required levels for each specification.

The Pooling Problem: Other Constraints



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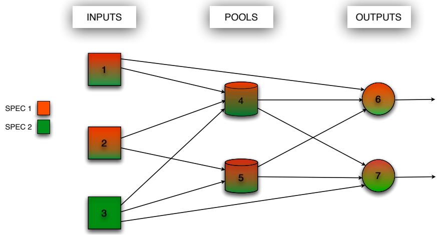

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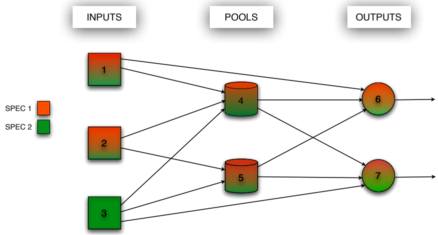

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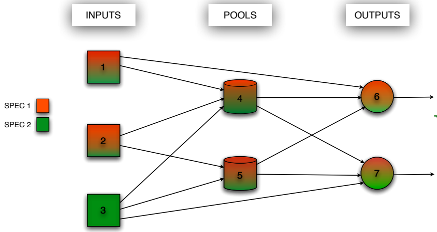
Tracking Specification



Data:

- λ_i^k: The value of specification k at input node i.

Variable:

- p_l^k: The value of specification k at node l
- y_{ab}: Flow along the arc (ab).

Specification Tracking: $$\underbrace{\sum_{i\in I} \lambda_i^k y_{il}}_{\text{Inflow of Spec k}} = \underbrace{p_l^k \left(\sum_{j\in J} y_{lj}\right)}_{\text{Out flow of Spec k}}$$

The pooling problem: 'P' formulation

[Haverly (1978)]

$$\max \quad \sum_{ij \in \mathcal{A}} w_{ij} y_{ij} \quad \text{(Maximize profit due to flow)}$$

Subject To:

1. Node and arc capacities.
2. Total flow balance at each node.
3. Specification balance at each pool.

$$\boxed{\sum_{i \in I} \lambda_i^k y_{il} = p_l^k \left(\sum_{j \in J} y_{lj} \right)} \quad < - - - \text{Write McCormick relaxation of these}$$

4. Bounds on p_j^k for all out put nodes j and specification k.

Q Model

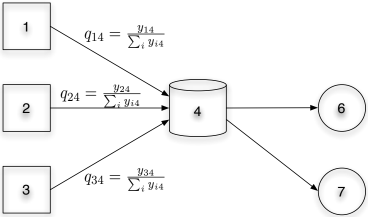
[Ben-Tal, Eiger, Gershovitz (1994)]

New Variable:

- q_{il} : fraction of flow to l from $i \in I$

$$\sum_{i \in I} q_{il} = 1, q_{il} \geq 0, i \in I.$$

  - $p_l^k = \sum_{i \in I} \lambda_i^k q_{il}$
- v_{ilj} : flow from input node i to output node j via pool node l.
  - $v_{ilj} = q_{il} y_{lj}$



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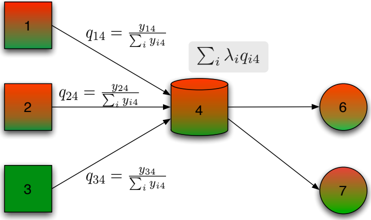

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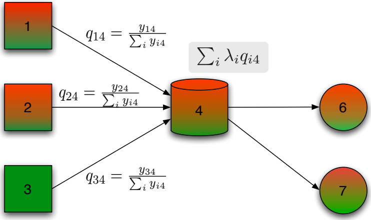

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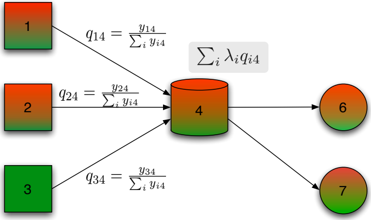


Q Model

$$\max \quad \sum_{i\in I, j\in J} w_{ij} y_{ij} + \sum_{i\in I, l\in L, j\in J} (w_{il} + w_{lj}) v_{ilj}$$

$$\text{s.t.} \quad v_{ilj} = q_{il} y_{lj} \;\; \forall i \in I, l \in L, j \in J <---\text{Write McCormick relaxation of these}$$

$$\sum_{i\in I} q_{il} = 1 \;\; \forall l \in L$$

$$a_j^k \left(\sum_{i\in I} y_{ij} + \sum_{l\in L} y_{lj} \right) \leq \sum_{i\in I} \lambda_i^k y_{ij} + \sum_{i\in I, l\in L} \lambda_i^k v_{ilj} \leq b_j^k \left(\sum_{i\in I} y_{ij} + \sum_{l\in L} y_{lj} \right)$$

Capacity constraints

All variables are non-negative

"PQ Model" Improved: Significantly better bounds

[Quesada and Grossmann (1995)], [Tawarmalani and Sahinidis (2002)]

$$\max \quad \sum_{i\in I, j\in J} w_{ij} y_{ij} + \sum_{i\in I, l\in L, j\in J} (w_{il} + w_{lj}) v_{ilj}$$

$$\text{s.t.} \quad v_{ilj} = q_{il} y_{lj} \ \forall i \in I, l \in L, j \in J <---\text{Write McCormick relaxation of these}$$

$$\sum_{i\in I} q_{il} = 1 \ \forall l \in L$$

$$a_j^k \left(\sum_{i\in I} y_{ij} + \sum_{l\in L} y_{lj} \right) \le \sum_{i\in I} \lambda_i^k y_{ij} + \sum_{i\in I, l\in L} \lambda_i^k v_{ilj} \le b_j^k \left(\sum_{i\in I} y_{ij} + \sum_{l\in L} y_{lj} \right)$$

Capacity constraints

All variables are non-negative

$$\sum_{i\in I} v_{ilj} = y_{lj} \ \forall l \in L, j \in J$$

$$\sum_{j\in J} v_{ilj} \le c_l q_{il} \ \forall i \in I, l \in L.$$

4.3
A covering-type bilinear knapsack set

A covering-type bilinear knapsack set

Consider the following set:

$$P := \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \mid \sum_{i=1}^{n} a_i \tilde{x}_i \tilde{y}_i \geq b\},$$

where $a_i \geq 0$ for all $i \in [n]$ and $b > 0$.

Note that this is an unbounded set.
For convenience of analysis consider rescaled version:

$$\boxed{P := \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i y_i \geq 1\},}$$

(For example: $x_i = \frac{a_i}{b}\tilde{x}_i, y_i = \tilde{y}_i$)

Is re-scaling okay?

Observation: Affine bijective map "commutes" with convex hull operation

Let $S \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be an affine bijective map. Then:

$$f(\mathrm{conv}(S)) = \mathrm{conv}(f(S)).$$

Proof

$$\begin{aligned} x \in f(\mathrm{conv}(S)) \quad &\iff \quad \exists y : x = f(y), y = \sum_{i=1} y^i \lambda_i, \lambda \in \Delta \\ &\iff \quad \exists y : x = f(y), f(y) = \sum_{i=1} f(y^i)\lambda_i, \lambda \in \Delta \ (f \text{ is bij. affine}) \\ &\iff \quad x \in \mathrm{conv}(f(S)). \end{aligned}$$

Careful: Not usually true if f is only bijective, but not affine!

The convex-hull of covering-type bilinear set

Theorem (Tawarmalani, Richard, Chung (2010))

Let $P := \{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i y_i \geq 1\}$. *Then*

$$\text{conv}(P) := \left\{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n \sqrt{x_i y_i} \geq 1\right\}.$$

Note: $\sum_{i=1}^n \sqrt{x_i y_i} \geq 1$ is a convex set because:

- $\sqrt{x_i y_i}$ is a concave function for $x_i, y_i \geq 0$.
- So $\sum_{i=1}^n \sqrt{x_i y_i}$ is a concave function.
- $f(x_i, y_i) := \sqrt{x_i y_i}$ is a positively-homogenous, i.e. $f(\eta(u,v)) = \eta f(u,v)$ for all $\eta > 0$.

Proof of Theorem: "⊆"

$$P := \left\{ (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^n x_i y_i \geq 1 \right\}.$$

$$\text{conv}(P) \underbrace{=}_{\text{To prove}} \underbrace{\left\{ (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^n \sqrt{x_i y_i} \geq 1 \right\}}_{H}.$$

$\text{conv}(P) \subseteq H$

- Sufficient to prove $P \subseteq H$. Let $(\hat{x}, \hat{y}) \in P$. Two cases:
  - If $\exists i$ such that $\hat{x}_i \hat{y}_i \geq 1$. Then $\sqrt{\hat{x}_i \hat{y}_i} \geq 1$ and thus $(\hat{x}, \hat{y}) \in H$.
  - Else $\hat{x}_i \hat{y}_i \leq 1$ for $i \in [n]$. Thus $\sum_{i=1}^n \sqrt{\hat{x}_i \hat{y}_i} \geq \sum_{i=1}^n \hat{x}_i \hat{y}_i \geq 1$ and thus $(\hat{x}, \hat{y}) \in H$.

Proof of Theorem: "$\supseteq$"

$\text{conv}(P) \supseteq H$

- Let $(\hat{x}, \hat{y}) := (\hat{x}_1, \hat{y}_1, \hat{x}_2, \hat{y}_2, \dots, \hat{x}_n, \hat{y}_n) \in H$. "WLOG:"

$$\Big(\underbrace{\hat{x}_1, \hat{y}_1}_{\sqrt{\hat{x}_1\hat{y}_1}=\lambda_1>0}, \underbrace{\hat{x}_2, \hat{y}_2}_{\sqrt{\hat{x}_2\hat{y}_2}=\lambda_2>0}, \underbrace{\hat{x}_3, \hat{y}_3}_{\sqrt{\hat{x}_3\hat{y}_3}=\lambda_3>0}, \underbrace{\hat{x}_4, \hat{y}_4}_{\hat{x}_4>0, \hat{y}_4=0} \dots, \underbrace{\hat{x}_n, \hat{y}_n}_{\hat{x}_n=0, \hat{y}_n>0}\Big)$$

- So we have $\lambda_1 + \lambda_2 + \lambda_3 \geq 1$. Let $\breve{\lambda}_i = \frac{\lambda_i}{\lambda_1+\lambda_2+\lambda_3}$ $\forall\ i \in [3]$.
- Consider the three points:

$$\begin{array}{lllllll} p^1 & := & \left(\frac{\hat{x}_1}{\breve{\lambda}_1}, \frac{\hat{y}_1}{\breve{\lambda}_1}, \right. & 0, 0, & 0, 0, & \frac{\hat{x}_4}{\breve{\lambda}_1}, 0, & \dots, & \left. 0, \frac{\hat{y}_n}{\breve{\lambda}_1}\right) \\ p^2 & := & (0, 0, & \frac{\hat{x}_2}{\breve{\lambda}_2}, \frac{\hat{y}_2}{\breve{\lambda}_2}, & 0, 0, & 0, 0, & \dots, & 0, 0) \\ p^3 & := & (0, 0, & 0, 0, & \frac{\hat{x}_3}{\breve{\lambda}_3}, \frac{\hat{y}_3}{\breve{\lambda}_3}, & 0, 0, & \dots, & 0, 0) \end{array}$$

- Trivial to verify that $\breve{\lambda}_1 p^1 + \breve{\lambda}_2 p^2 + \breve{\lambda}_3 p^3 = (\hat{x}, \hat{y})$, and $\breve{\lambda}_1 + \breve{\lambda}_2 + \breve{\lambda}_3 = 1$.
- $\boxed{\frac{\hat{x}_1}{\breve{\lambda}_1} \cdot \frac{\hat{y}_1}{\breve{\lambda}_1} = \left(\frac{\sqrt{\hat{x}_i\hat{y}_i}}{\breve{\lambda}_1}\right)^2 = \left(\frac{\lambda_1}{\breve{\lambda}_1}\right)^2 \geq 1 \Rightarrow p^1 \in P.}$ Similarly $p^2 \in P, p^3 \in P$.

An interpretation of the proof

The result in [Tawarmalani, Richard, Chung (2010)] is more general.

"Two ingredients" in the proof

- "Orthogonal disjunction": Define $P_i := \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \,|\, x_i y_i \geq 1\}$. Then it can be verified that:

$$\text{conv}(P) = \text{conv}\left(\bigcup_{i=1}^{n} P_i\right).$$

- Positive homogenity: P_i is convex set. Also,

  $P_i := \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \,|\, \sqrt{x_i y_i} \geq 1\}$ < ––The "correct way" to write the set

  This single term convex hull is described using the positive homogenous function.

Another example of convexification from [Tawarmalani, Richard, Chung (2010)]

Example

$S := \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6_+ \,|\, x_1x_2x_3 + x_4x_5 + x_6 \geq 1\}$, then

$$\text{conv}(S) := \left\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6_+ \,\middle|\, (x_1x_2x_3)^{\frac{1}{3}} + (x_4x_5)^{\frac{1}{2}} + x_6 \geq 1\right\}$$

Lets talk about "representability" of the convex hull

- Up till now, we had polyhedral convex hull. This bilinear covering set yields our first non-polyhedral example of convex hull.
- It turns out the set:

$$\left\{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^{n} \sqrt{x_i y_i} \geq 1\right\}$$

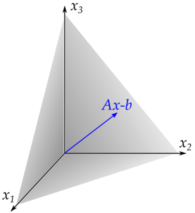
is second order cone representable (SOCr).

A quick review of second order cone representable sets: Introduction

Polyhedron:

$$Ax - b \in \mathbb{R}^m_+$$
$$x \in \mathbb{R}^n$$

$\mathbb{R}^m_+$ is a closed, convex, pointed and full dimensional cone.



Conic set:

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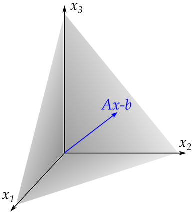

Conic set:

Second order conic representable set

Conic set

$$Ax - b \in K$$

Definiton: Second order cone

$$K := \{u \in \mathbb{R}^m \mid \|(u_1, \ldots, u_{m-1})\|_2 \leq u_m\}$$

Second order conic representable (SOCr) set

A set $S \subseteq \mathbb{R}^n$ is a second order cone representable if,

$$S := \text{Proj}_x \{(x, y) \mid Ax + Gy - b \in (K_1 \times K_2 \times K_3 \times \cdots \times K_p)\},$$

where K_i's are second order cone. Or equivalently,

$$S := \text{Proj}_x\{(x, y) \mid \|A^i x + G^i y - b^i\|_2 \leq A^{i0} x + G^{i0} y - b^{i0} \;\; \forall i \in [p]\},$$

Lets get back to our convex hull

$$\left\{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^{n} \sqrt{x_i y_i} \geq 1 \right\}$$

- In fact, the above set is Second order cone (SOCr) representable:

$$\begin{aligned} x, y &\in \mathbb{R}^n_+ \\ \sum_{i=1}^{n} u_i &\geq 1 \\ \sqrt{x_i y_i} &\geq u_i \;\; \forall i \in [n] \end{aligned}$$

Lets get back to our convex hull

$$\left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} \sqrt{x_i y_i} \geq 1 \right\}$$

- In fact, the above set is Second order cone (SOCr) representable:

$$\begin{aligned} x, y &\in \mathbb{R}^n_+ \\ \sum_{i=1}^{n} u_i &\geq 1 \\ x_i y_i &\geq u_i^2 \;\; \forall i \in [n] \end{aligned}$$

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- In fact, the above set is Second order cone (SOCr) representable:

$$\begin{aligned} x, y &\in \mathbb{R}^n_+ \\ \sum_{i=1}^{n} u_i &\geq 1 \\ (x_i + y_i)^2 - (x_i - y_i)^2 &\geq 4u_i^2 \;\; \forall i \in [n] \end{aligned}$$

Lets get back to our convex hull

$$\left\{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n \sqrt{x_i y_i} \geq 1\right\}$$

- In fact, the above set is Second order cone (SOCr) representable:

$$\begin{aligned} x, y &\in \mathbb{R}^n_+ \\ \sum_{i=1}^n u_i &\geq 1 \\ x_i + y_i &\geq \sqrt{(2u_i)^2 + (x_i - y_i)^2} \ \forall i \in [n] \end{aligned}$$

Our convex hull is SOCr

$$\left\{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^{n} \sqrt{x_i y_i} \geq 1 \right\}$$

- In fact, the above set is Second order cone (SOCr) representable:

$$\begin{aligned} x, y &\in \mathbb{R}^n_+ \\ \sum_{i=1}^{n} u_i &\geq 1 \\ (x_i + y_i) &\geq \left\| \begin{matrix} 2u_i \\ (x_i - y_i) \end{matrix} \right\|_2 \quad \forall i \in [n] \end{aligned}$$

Our convex hull is SOCr

$$\left\{(x,y)\in\mathbb{R}^n_+\times\mathbb{R}^n_+\,\middle|\,\sum_{i=1}^n\sqrt{x_iy_i}\geq 1\right\}$$

- In fact, the above set is Second order cone (SOCr) representable:

$$\begin{aligned}
x_i &\geq \|0\|_2 \forall i\in[n]\\
y_i &\geq \|0\|_2 \forall i\in[n]\\
\sum_{i=1}^n u_i - 1 &\geq \|0\|_2\\
(x_i+y_i) &\geq \left\|\begin{matrix}2u_i\\(x_i-y_i)\end{matrix}\right\|_2 \ \forall i\in[n]
\end{aligned}$$

5

Convex hull of a general one-constraint quadratic constraint

Our next goal

Theorem (Santana, D. (2019))

Let

$$S := \{x \in \mathbb{R}^n \mid x^\top Q x + \alpha^\top x = g, \; x \in P\}, \tag{2}$$

where $Q \in \mathbb{R}^{n \times n}$ *is a symmetric matrix,* $\alpha \in \mathbb{R}^n$, $g \in \mathbb{R}$ *and* $P := \{x \mid Ax \leq b\}$ *is a polytope. Then* $\mathrm{conv}(S)$ *is second order cone representable.*

- The proof is contructive. So in principle, we can build the convex hull using the proof.
- The size of the second order “extended formulation” is exponential in size.
- The result holds if we replace the quadratic equation with an inequality.

Main ingredients to proof theorem

Basically 3 ingredients:

- Hillestad-Jacobsen Theorem on reverse convex sets.
- Richard-Tawarmalani lemma for continuous function.
- Convex hull of union of conic sets.

5.1

Reverse convex sets

A common structure

$$S := P \setminus \left(\bigcup_{i=1}^{m} \text{int}(C^i) \right),$$

where P is a polyope and C^i's are closed convex sets.

- Where have we seen this before in context of integer programming? When $m = 1$: Intersection cuts!
- Note that $\text{conv}(P \setminus C)$ is a polytope!

A common structure

$$S := P \Big\backslash \left(\bigcup_{i=1}^{m} \text{int}(C^i) \right),$$

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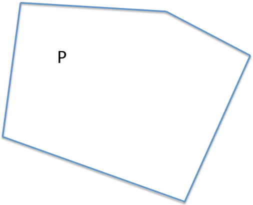

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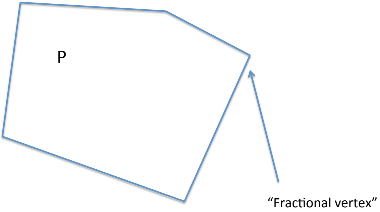

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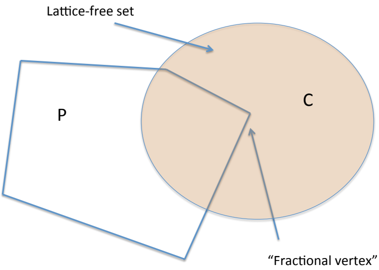

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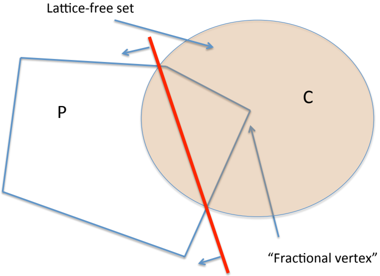

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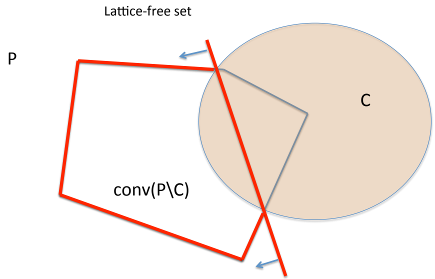
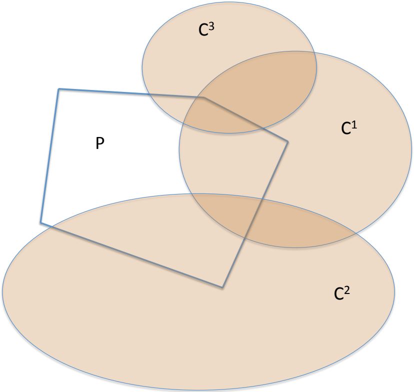


$m \geq 2$



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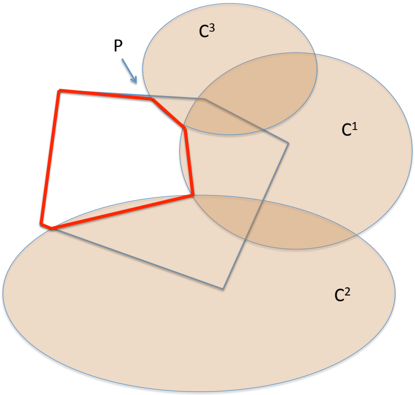


Do we have a theorem?

Theorem (Hillestad, Jacobsen (1980))

Let $P \subseteq \mathbb{R}^n$ be a polytope and let $C^1, \dots, C^m$ be closed convex sets. Then

$$\text{conv}\left(P \Big\backslash \left(\bigcup_{i=1}^{m} \text{int}(C^i)\right)\right)$$

is a polytope.

The proof is again going to use the Krein-Milman Theorem. In particular, we will prove that $S = P \backslash \left(\bigcup_{i=1}^{m} \text{int}(C^i)\right)$ has a finite number of extreme points.

A key Lemma

Necesary condition for extreme points of S

Let

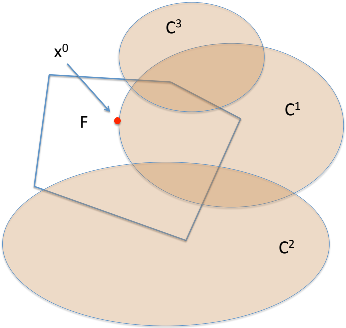
$$S := P \left\backslash \left(\bigcup_{i=1}^{m} \text{int}(C^i) \right) \right. ,$$

where P is a polyope and C^i's are closed convex sets.
Let F be a face of P of dimension d. Let $x^0 \in \text{rel.int}(F)$ be an extreme point of S. Then x^0 belongs to the boundary of at least d of the convex sets C^is.

Proof of Lemma

Application of separation theorem for convex set

- Assume by contradiction: $x^0 \in \text{rel.int}(F)$ and $x^0 \in \text{bnd}(C^i)$ for $i \in [k]$ where $k < d$.
- Let $(a^i)^\top x \leq b^i$ be a separating hyperplane between x^0 and $\text{int}(C^i)$ for $i \in [k]$. Let $V := \{x \,|\, (a^i)^\top x = b^i \;\; i \in [k]\}$
- Since $\dim(F) = d$ and $\dim(V) \geq n - k$, we have $\dim(\text{aff.hull}(F) \cap V) \geq d - k \geq 1$.



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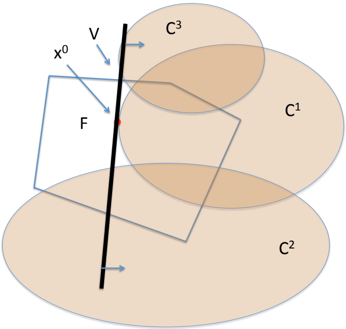

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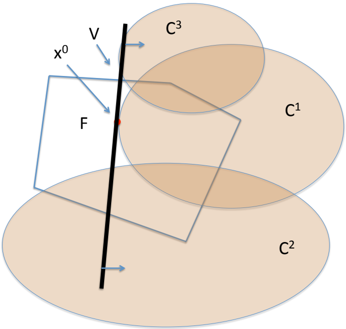
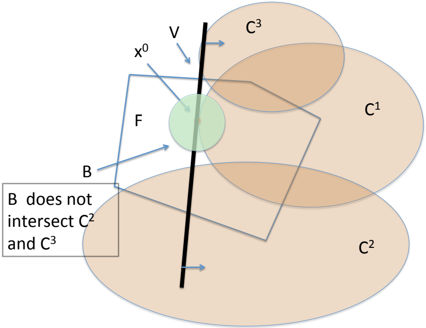


Proof of Lemma

Application of separation theorem for convex set

- Also there is a ball B, centered at x^0, such that (i) $B \cap \text{aff.hull}(F) \subseteq F$, (ii) $B \cap C_i = \emptyset \; i \in \{k+1, \dots, m\}$.
- Then, $B \cap (\text{aff.hull}(F) \cap V) \subseteq F \setminus \bigcup_{i=1}^m \text{int}(C^i)$ and $\dim(B \cap (\text{aff.hull}(F) \cap V)) \geq 1$.
- So x^0 is not an extreme point in S.



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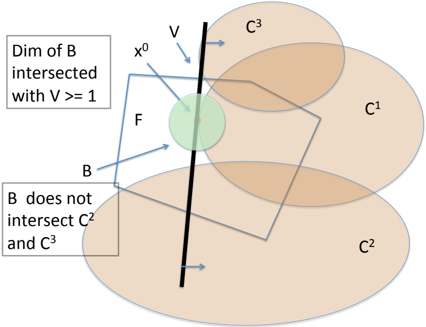

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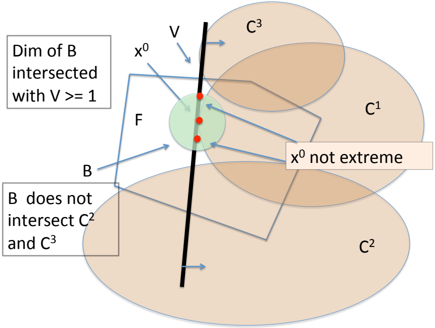
- So x^0 is not an extreme point in S.



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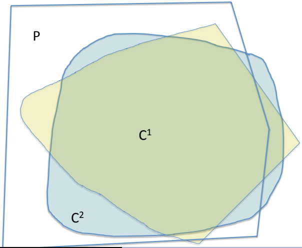

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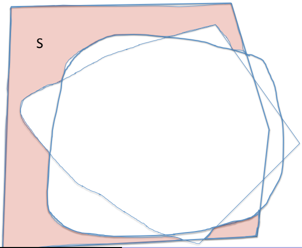

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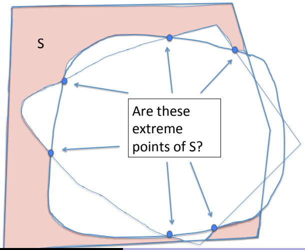

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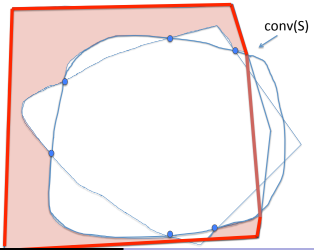


One more idea to prove theorem

Dominating pattern

Let $x^1, x^2 \in S$. We say that the pattern of x^2 dominates the pattern of x^1 if:

1. x^1 and x^2 belong to the relative interior of the same face F of P
2. If $x^1 \in \text{bnd}(C_j)$, then $x^2 \in \text{bnd}(C_j)$.

Another lemma

Lemma

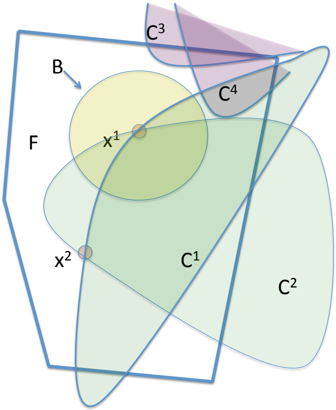
Let $x^1, x^2 \in S$ be distinct points. If the pattern of x^2 dominates the pattern of x^1, then x^1 is not an extreme point of S.

This lemma completes the proof of the Theorem:

- We want to prove total number of extreme points in finite.
- Lemma 1 tell us that for an extreme point, which is in rel.int of a face F of dim d, it must be on the boundary of d convex sets.
- For any face and any "pattern" of convex sets, there can only be one extreme point of S. Thus, the number of extreme points of S is finite.

Proof of Lemma 2

- x^2 dominates x^1.
- WLOG let $x^1, x^2 \in \text{bnd}(C^i)$ for $i \in [k]$ and there is a ball B centered around x^2 such that (i) $B \cap \text{aff.hull}(F) \subseteq F$ and (ii) $B \cap C^j = \varnothing$ for $j \in \{k+1, \dots, m\}$.
- Consider $x^0 \in B$ such that x^2 is a convex combination of x^1 and x^0. It remains to show $x^0 \in S$:
  - Clearly $x^0 \in F \subseteq P$.
  - $B \cap C^j = \varnothing \Rightarrow x^0 \notin C^j \; \{k+1, \dots, m\}$.
  - Suppose $x^0 \in \text{int}(C^j)$ for $j \in [k]$, by dominance $x^2 \in C^j$, then $x^2 \in \text{int}(C^j)$, a contradiction. So $x^0 \notin \text{int}(C^j)$ for $j \in [k]$.



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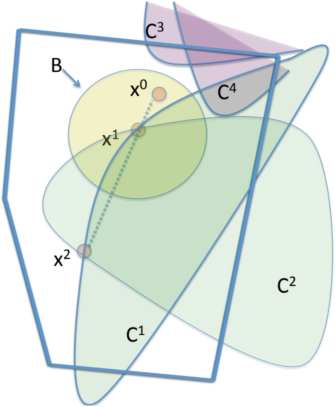

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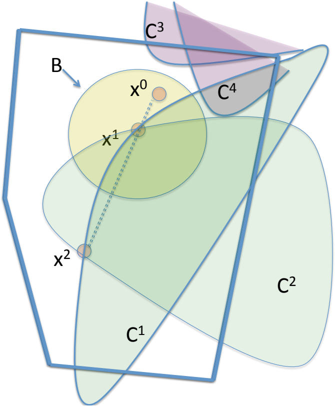

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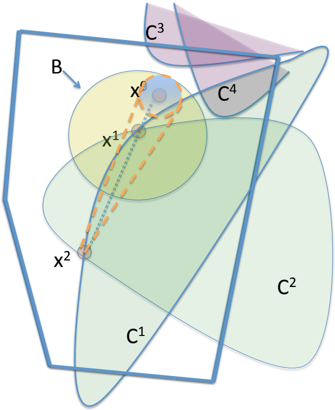

5.2
Dealing with "equality sets": The Richard-Tawamalani Lemma

The Richard-Tawarmalani Lemma

Lemma (Richard Tawarmalani (2014))

Consider the set $S := \{x \in \mathbb{R}^n \,|\, f(x) = 0, x \in P\}$ *where* f *is a continuous function and* P *is a convex set. Then:*

$$\text{conv}(S) = \text{conv}\left(S^{\leq}\right) \bigcap \text{conv}\left(S^{\geq}\right),$$

where

$$\begin{aligned} S^{\leq} &:= \{x \in \mathbb{R}^n \,|\, f(x) \leq 0, x \in P\} \\ S^{\geq} &:= \{x \in \mathbb{R}^n \,|\, f(x) \geq 0, x \in P\} \end{aligned}$$

Proof of Lemma

- Clearly
$$\text{conv}(S) \subseteq \text{conv}\left(S^{\leq}\right) \bigcap \text{conv}\left(S^{\geq}\right)$$
- So it is sufficient to prove
$$\text{conv}(S) \supseteq \text{conv}\left(S^{\leq}\right) \bigcap \text{conv}\left(S^{\geq}\right)$$
- Pick $x^0 \in \text{conv}\left(S^{\leq}\right) \bigcap \text{conv}\left(S^{\geq}\right)$, we need to show $x^0 \in \text{conv}(S)$.

Claim 1

Claim: $x^0 \in \text{conv}\left(S^{\leq}\right)$ implies x^0 can be written as convex combination of points in S and at most one point from $S^{\leq} \setminus S$.

Proof

- Suppose $x^0 = \sum_{i=1}^{n+1} \lambda_i y^i$, $\lambda \in \Delta$, where $y^i \in S$
- Suppose WLOG, $y^1, y^2 \in S^{\leq} \setminus S$. Two cases:
  - $y^0 := \frac{1}{\lambda_1 + \lambda_2}(\lambda_1 y^1 + \lambda_2 y^2) \in S^{\leq}$: In this case replace the two points y^1 and y^2 by the point y^0 and we have one less point from $S^{\leq} \setminus S$ whose convex combination gives x^0.

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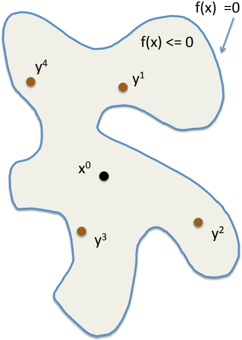

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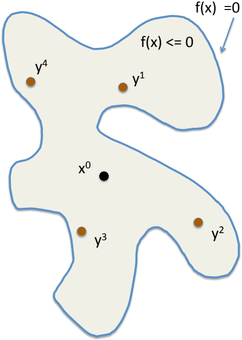

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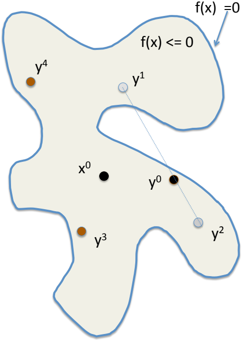

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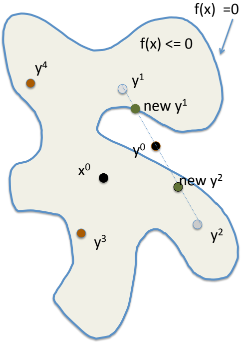

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- Repeat above argument finite number of times to arrive at Claim.

Claim 1

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- Repeat above argument finite number of times to arrive at Claim.

Completing proof of Lemma

- Remember, for $x^0 \in \text{conv}\,(S^{\leq}) \bigcap \text{conv}\,(S^{\geq})$, we need to show $x^0 \in \text{conv}(S)$.
- From previous claim applied to $S^{\leq}$ and $S^{\geq}$:

$$x^0 = \lambda_0 y^0 + \sum_{i=1}^{n} \lambda_i y^i, \;\; \lambda \in \Delta, y^0 \in S^{\leq}, y^i \in S \;\; i \geq 1 \tag{3}$$

$$x^0 = \mu_0 w^0 + \sum_{i=1}^{n} \mu_i w^i, \;\; \mu \in \Delta, w^0 \in S^{\geq}, w^i \in S \;\; i \geq 1. \tag{4}$$

- (Again) by intermediate value theorem, suppose $z^0 := \gamma y^0 + (1-\gamma) w^0$ satisfies $z^0 \in S$ for $\gamma \in [0,1]$. Then by taking suitable convex combination of (3) and (4), $\exists \delta \in \Delta$

$$\delta_0 z^0 + \sum_{i=1}^{2} \delta_i y^i + \sum_{i=n+1}^{2n} \delta_i w^{i-n} = x^0, \;\; \lambda \in \Delta, z^0, y^i, w^i \in S \;\; i \geq 1.$$

An important corollary

Theorem (Hillestad, Jacobsen (1980))

Let $P \subseteq \mathbb{R}^n$ be a polytope and let $C^1, \dots C^m$ be closed convex sets. Then

$$\text{conv}\left(P \bigg\backslash \left(\bigcup_{i=1}^{m} \text{int}(C^i)\right)\right)$$

is a polytope.

Lemma (Richard Tawarmalani (2014))

Consider the set $S := \{x \in \mathbb{R}^n \,|\, f(x) = 0, x \in P\}$ where f is a continuous function and P is a convex set. Then:

$$\text{conv}(S) = \text{conv}\left(S^{\leq}\right) \bigcap \text{conv}\left(S^{\geq}\right),$$

where

$$\begin{aligned} S^{\leq} &:= \{x \in \mathbb{R}^n \,|\, f(x) \leq 0, x \in P\} \\ S^{\geq} &:= \{x \in \mathbb{R}^n \,|\, f(x) \geq 0, x \in P\} \end{aligned}$$

An important corollary: The SOCr-Boundary Corollary

Corollary

Let $S := \{x \in P \mid f(x) = 0\}$ such that

- $f : \mathbb{R}^n \to \mathbb{R}$ is real-valued convex function such that $\{x \mid f(x) \leq 0\}$ is SOCr.
- $P \subseteq \mathbb{R}^n$ is a polytope.

Then $\text{conv}(S)$ is SOCr.

Proof

- Convexity implies continuity of f, so by the Richard-Tawarmalani Lemma, $\text{conv}(S) = \text{conv}(S^{\leq}) \cap \text{conv}(S^{\geq})$.
- $\text{conv}(S^{\leq}) = \{x \in P \mid f(x) \leq 0\} = \underbrace{\{x \mid f(x) \leq 0\} \cap P}_{\text{SOCr}}$.
- $\text{conv}(S^{\geq}) = \underbrace{\{x \in P \mid f(x) \geq 0\}}_{\equiv P \setminus \text{int}(\{x \mid f(x) \leq 0\}}$), so $\text{conv}(S^{\geq})$ is a polytope by the Hillestad-Jacobsen Theorem. A polytope is a SOCr representable.

An important corollary: The SOCr-Boundary Corollary

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Let $S := \{x \in P \,|\, f(x) = 0\}$ such that

- $f : \mathbb{R}^n \to \mathbb{R}$ is real-valued convex function such that $\{x \,|\, f(x) \leq 0\}$ is SOCr.
- $P \subseteq \mathbb{R}^n$ is a polytope.

Then conv(S) is SOCr.

If T is boundary of a SOCr set, then convex hull of T interesected with a polytope is SOCr.

5.3
Ingredient 3: Convex hull of union of conic sets

Ingredient - Convex hull of union of conic sets

Theorem

Let $P^1 := \{x \in \mathbb{R}^n \,|\, A^1 x - b^1 \in K^1\}$ and $P^2 := \{x \in \mathbb{R}^n \,|\, A^2 x - b^2 \in K^2\}$ be bounded conic sets. Then

$$\text{conv}(P^1 \bigcup P^2) = \text{Proj}_x \underbrace{\left\{ \begin{pmatrix} x \in \mathbb{R}^n, \\ x^1 \in \mathbb{R}^n, \\ x^2 \in \mathbb{R}^n, \\ \lambda \in \mathbb{R} \end{pmatrix} \middle| \begin{array}{l} A^1 x^1 - b^1 \lambda \in K^1, \\ A^2 x^2 - b^2 (1 - \lambda) \in K^2, \\ x = x^1 + x^2, \\ \lambda \in [0, 1] \end{array} \right\}}_{Q}$$

Corollary for SOCr sets

Let S^1 and S^2 be two bounded SOCr sets. Then $\text{conv}(S^1 \cup S^2)$ is also SOCr.

Proof: $\text{conv}(P^1 \cup P^2) \subseteq \text{Proj}_x(Q)$ inclusion

$$Q := \left\{ \begin{pmatrix} x \in \mathbb{R}^n, \\ x^1 \in \mathbb{R}^n, \\ x^2 \in \mathbb{R}^n, \\ \lambda \in \mathbb{R} \end{pmatrix} \middle| \begin{array}{l} A^1x^1 - b^1\lambda \in K^1, \\ A^2x^2 - b^2(1-\lambda) \in K^2, \\ x = x^1 + x^2, \\ \lambda \in [0,1] \end{array} \right\}$$

$\text{conv}(P^1 \cup P^2) \subseteq \text{Proj}_x(Q)$

- If $\tilde{x} \in P^1$, then $\tilde{x} \in \text{Proj}_x(Q)$ (by setting $x = x^1 = \tilde{x},\ x^2 = 0,\ \lambda = 1$).
- Similarly if $\tilde{x} \in P^2$, then $\tilde{x} \in \text{Proj}_x(Q)$.
- $P^1 \cup P^2 \subseteq \text{Proj}_x(Q)$
- $\text{conv}(P^1 \cup P^2) \subseteq \text{Proj}_x(Q)$ (Because $\text{Proj}_x(Q)$ is a convex set)

Proof: $\text{conv}(P^1 \bigcup P^2) \supseteq \text{Proj}_x(Q)$ inclusion

$$Q := \left\{ \begin{pmatrix} x \in \mathbb{R}^n, \\ x^1 \in \mathbb{R}^n, \\ x^2 \in \mathbb{R}^n, \\ \lambda \in \mathbb{R} \end{pmatrix} \middle| \begin{array}{l} A^1x^1 - b^1\lambda \in K^1, \\ A^2x^2 - b^2(1-\lambda) \in K^2, \\ x = x^1 + x^2, \\ \lambda \in [0,1] \end{array} \right\}$$

Let $\tilde{x}, \tilde{x}^1, \tilde{x}^2, \tilde{\lambda} \in Q$.

Case 1: $0 < \tilde{\lambda} < 1$

- $$K^1 \underbrace{\ni}_{K^1 \text{ is a cone}} \frac{1}{\tilde{\lambda}} \underbrace{\left(A^1\tilde{x}^1 - \tilde{\lambda}b^1\right)}_{\in K^1} = A^1\left(\frac{\tilde{x}^1}{\tilde{\lambda}}\right) - b^1$$
- So $\left(\frac{\tilde{x}^1}{\tilde{\lambda}}\right) \in P^1$.
- Similarly: $\frac{\tilde{x}^2}{1-\tilde{\lambda}} \in P^2$.
- Also $\tilde{x} = \tilde{\lambda} \cdot \left(\frac{\tilde{x}^1}{\tilde{\lambda}}\right) + (1-\tilde{\lambda}) \cdot \frac{\tilde{x}^2}{1-\tilde{\lambda}}$.
- So $\tilde{x} \in \text{conv}(P^1 \bigcup P^2)$.

Proof: $\mathrm{conv}(P^1 \cup P^2) \supseteq \mathrm{Proj}_x(Q)$ inclusion

$$Q := \left\{ \begin{pmatrix} x \in \mathbb{R}^n, \\ x^1 \in \mathbb{R}^n, \\ x^2 \in \mathbb{R}^n, \\ \lambda \in \mathbb{R} \end{pmatrix} \middle| \begin{array}{l} A^1 x^1 - b^1 \lambda \in K^1, \\ A^2 x^2 - b^2(1-\lambda) \in K^2, \\ x = x^1 + x^2, \\ \lambda \in [0,1] \end{array} \right\}$$

Let $\tilde{x}, \tilde{x}^1, \tilde{x}^2, \tilde{\lambda} \in Q$.

Case 2: $\tilde{\lambda} = 1$

- $\tilde{x}^1 \in P^1$, since $A^1\tilde{x}^1 - b^1 \cdot 1 \in K^1$.
- Claim: $\tilde{x}^2 = 0$: Note $A^2\tilde{x}^2 = 0$. If $\tilde{x}^2 \neq 0$, then for any $x^0 \in P^2$, we have that for any $M > 0$, $A^2(x^0 + M\tilde{x}^2) - b^2 = MA^2\tilde{x}^2 + A^2(x^0) - b^2 = A^2x^0 - b^2 \in K^2$. So $x^0 + M\tilde{x}^2 \in P^2$ for $M > 0$, i.e., P^2 is unbounded, a contradition.
- So $\tilde{x} = \tilde{x}^1 \in P^1 \subseteq \mathrm{conv}(P^1 \cup P^2)$.

Case 3: $\tilde{\lambda} = 0$

Same as previous case

5.4
Proof of one-row-theorem

One row theorem

Theorem (Santana, D. (2019))

Let

$$S := \{x \in \mathbb{R}^n \mid x^\top Q x + \alpha^\top x = g,\ x \in P\}, \tag{5}$$

where $Q \in \mathbb{R}^{n \times n}$ *is a symmetric matrix,* $\alpha \in \mathbb{R}^n$, $g \in \mathbb{R}$ *and* $P := \{x \mid Ax \leq b\}$ *is a polytope. Then* $\operatorname{conv}(S)$ *is second order cone representable.*

Proof of Thm: Basic building block

- Krein-Milman Theorem: If S is compact, $\text{conv}(S) = \text{conv}(\text{ext}(S))$.
- If $\text{ext}(S) \subseteq \bigcup_{k=1}^{m} T_k \subseteq S$, then

$$\boxed{\text{conv}(S) = \text{conv}\left(\bigcup_{k=1}^{m} \text{conv}(T_k)\right)}$$

- Finally, if $\text{conv}(T_k)$ is SOCr, then $\text{conv}(S)$ is SOCr.

Structure Lemma on Quadratic functions

Lemma

Consider a set defined by a single quadratic equation. Then exactly one of the following occurs:

1. ***Case 1:*** *It is the boundary of a SOCP representable convex set,*
2. ***Case 2:*** *It is the union of boundary of two disjoint SOCP representable convex set; or*
3. ***Case 3:*** *It has the property that, through every point, there exists a straight line that is entirely contained in the surface.*

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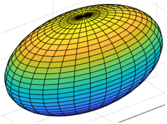

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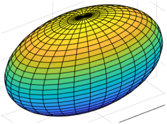

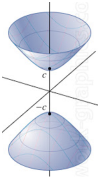

Structure Lemma on Quadratic functions

Lemma

*Consider a **set defined by a single quadratic equation**. Then exactly one of the following occurs:*

1. ***Case 1:*** *It is the boundary of a SOCP representable convex set,*
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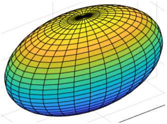

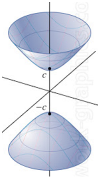

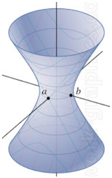

Ruled surface are beautiful!

Proof of Thm (sketch)

Using the Structure Lemma $S := \{x \in \mathbb{R}^n \,|\, x^\top Q x + \alpha^\top x = g, x \in P\}$

1. If in Case 1 or Case 2: (i.e., the boundry of SOCr convex set or union of boundary of two SOCr sets), then done!
   (Via SOCr-boundary Corollary; and Convex hull of union of SOCr sets Theorem)
2. Otherwise:
   1. Because of the lines (Case 3), no point in the relative interior of the polytope can be an extreme point;
   2. Intersect the quadratic with each facet of the polytope;
   3. Each intersection yields a new quadratic set of the same form, but in lower dimension;
3. Repeat above argument for each facet.

Basically: **(i) Consider all faces of P such that the quadratic on those faces are in Case 1 or Case 2. (ii) Then for these cases, write down the conv hull of the quadratic interested with the face– which is SOCr due to SOCr-boundary Corollary (iii) Take convex hull of the union of these SOCr set — which is SOCr due to the Convex hull of union of SOCr sets Theorem.**

Proof of Structure Lemma

Lemma: Proof of Structure Lemma — Reduction

Let T be a set defined by the a quadratic equation. If F is an affine bijective map, then:

1. T is Case1, Case 2, Case 3 iff $F(S)$ is in Case 1, Case 2, Case 3 (respectively)

Then, we rewrite

$$T := \{u \in \mathbb{R}^n \mid \; u^\top Q u + c^\top u = d\},$$

as

$$T = \Big\{(w, x, y) \in \mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}} \times \mathbb{R}^{n_l} \mid \sum_{i=1}^{n_{q+}} w_i^2 - \sum_{j=1}^{n_{q-}} x_j^2 + \sum_{k=1}^{n_l} y_k = d, \; \Big\},$$

where we may assume $d \geq 0$.

Proof of Structure Lemma

$$T = \Big\{(w, x, y) \in \mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}} \times \mathbb{R}^{n_l} \mid \sum_{i=1}^{n_{q+}} w_i^2 - \sum_{j=1}^{n_{q-}} x_j^2 + \sum_{k=1}^{n_l} y_k = d, \Big\}$$

Lemma

Assuming T as above and $d \geq 0$, we have:

Case	*Classification*
1) $n_l \geq 2$	*Case 3: straight line*
2) $n_{q+} \leq 1,\ n_l = 0$	*Case 1 or Case 2*
3) $n_{q+} n_{q-} = 0,\ n_l \leq 1$	*Case 1 or Case 2*
4) $n_{q+},\ n_{q-} \geq 1,\ n_l = 1$	*Case 3: straight line*
5) $n_{q+} \geq 2,\ n_{q-} \geq 1,\ n_l = 0$	*Case 3: straight line*

Proof of Structure Lemma

First four cases are straightforward.

Last case of previous lemma

$$T = \left\{(w, x) \in \mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}} \mid \sum_{i=1}^{n_{q+}} w_i^2 - \sum_{j=1}^{n_{q-}} x_j^2 = d, \right\},$$

where $d \geq 0$, $n_{q+} \geq 2$, and $n_{q-} \geq 1$. Then through every point in T, there exists a *straight* line that is *entirely* contained in T.

Proof of last case

Proof

- Consider a vector $(\hat{w}, \hat{x}) \in (\mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}}) \in T$.
- We want to show that there is a line $\{(\hat{w}, \hat{x}) + \lambda(u, v) \,|\, \lambda \in \mathbb{R}\}$ satisfies the quadratic equation of T, where $(u, v) \neq 0$. We consider the case when $(\hat{w}, \hat{x}) \neq 0$ [Other case trivial]:
- In this case $\hat{w} \neq 0$, since otherwise $-\sum_{j=1}^{n_{q-}} \hat{x}_j^2 = d \geq 0$ implies $\hat{x} = 0$. Then observe that:

$$\boxed{\sum_{i=1}^{n_{q+}} \hat{w}_i^2 = d + \sum_{j=1}^{n_{q-}} \hat{x}_j^2 \geq \hat{x}_1^2 \Leftrightarrow \frac{|\hat{x}_1|}{\|\hat{w}\|_2} \leq 1.}$$

$$\begin{aligned} d &= \sum_{i=1}^{n_{q+}} (\hat{w}_i + \lambda u_i)^2 - \sum_{i=1}^{n_{q-}} (\hat{x}_i + \lambda v_i)^2 \;\; \forall \lambda \in \mathbb{R} \\ \Leftrightarrow d &= \left(\sum_{i=1}^{n_{q+}} \hat{w}_i^2 - \sum_{i=1}^{n_{q-}} \hat{x}_i^2\right) + \lambda^2 \left(\sum_{i=1}^{n_{q+}} u_i^2 - \sum_{i=1}^{n_{q-}} v_i^2\right) + 2\lambda \left(\sum_{i=1}^{n_{q+}} \hat{w}_i u_i - \sum_{i=1}^{n_{q-}} \hat{x}_i v_i\right) \;\; \forall \lambda \in \mathbb{R} \end{aligned}$$

Proof of last case - contd.

$$\boxed{\frac{|\hat{x}_1|}{\|\hat{w}\|_2} \leq 1.}$$

$$\Leftrightarrow \sout{d} = \sout{\left(\sum_{i=1}^{n_{q+}} \hat{w}_i^2 - \sum_{i=1}^{n_{q-}} \hat{x}_i^2\right) +} \lambda^2 \left(\sum_{i=1}^{n_{q+}} u_i^2 - \sum_{i=1}^{n_{q-}} v_i^2\right) + 2\lambda \left(\sum_{i=1}^{n_{q+}} \hat{w}_i u_i - \sum_{i=1}^{n_{q-}} \hat{x}_i v_i\right) \ \forall \lambda \in \mathbb{R}$$

$$\Leftrightarrow \quad \sum_{i=1}^{n_{q+}} u_i^2 - \sum_{i=1}^{n_{q-}} v_i^2 = 0, \ \sum_{i=1}^{n_{q+}} \hat{w}_i u_i - \sum_{i=1}^{n_{q-}} \hat{x}_i v_i = 0. \quad (6)$$

- We set $v_1 = 1$ and $v_j = 0$ for all $j \in \{2, \dots, n_{q-}\}$. Then satisfying (6) is equivalent to finding real values of u satisfying:

$$\sum_{i=1}^{n_{q+}} u_i^2 = 1, \quad \sum_{i=1}^{n_{q+}} \hat{w}_i u_i = \hat{x}_1.$$

- This is the intersection of a circle of radius 1 in dimension two or higher (since $n_{q+} \geq 2$ in this case) and a hyperplane whose distance from the origin is $\frac{|\hat{x}_1|}{\|\hat{w}\|_2}$. Done!

Discussion

Classify: conv.hull of QCQP substructure is SOCr?

Is SOCP representable:

1. One quadratic equality (or inequality) constraint $\bigcap$ polytope.
2. Two quadratic inequalities ([Yıldıran (2009)], [Bienstock, Michalka (2014)], [Burer, Kılınç-Karzan (2017)], [Modaresi, Vielma (2017)])

Is not SOCP representable:

1. Already in 10 variables, 5 quadratic equalities, 4 quadratic inequalities, 3 linear inequalities ([Fawzi (2018)])

Other simple sets (with mostly SDP based convex hulls): highly incomplete literature review

- Related to study of generalized trust region problem:
$$\inf \quad x^\top Q^0 x + (A^0)^\top x \quad \text{s.t.} \quad x^\top Q^1 x + (A^1)^\top x + b^1 \le 0$$
[Fradkov and Yakubovich (1979)] showed SDP relaxation is tight. Since then work by: [Sturm, Zhang (2003)], [Ye, Zhang (2003)], [Beck, Eldar(2005)] [Burer, Anstreicher (2013)], [Jeyakumar, Li (2014)], [Yang, Burer (2015) (2016)], [Ho-Nguyen, Kılınç-Karzan (2017)], [Wang, Kln-Karzan (2019)]
- Explicit descriptions for the convex hull of the intersection of a single nonconvex quadratic region with other structured sets [Yıldıran (2009)], [Luo, Ma, So, Ye, Zhang (2010)], [Bienstock, Michalka (2014)], [Burer (2015)], [Kılınç-Karzan, Yıldız (2015)],[Yıldız, Cornuejols (2015)], [Burer and Kılınç-Karzan (2017)], [Yang, Anstreicher, Burer (2017)], [Modaresi and Vielma (2017)]
- SDP tight for general QCQPs? [Burer, Ye(2018)], [Wang, Kılınç-Karzan (2020)].
- Approximation Guarantees. [Nesterov (1997)], [Ye(1999)] [Ben-Tal, Nemirovski (2001)]

6

Back to convexification of functions: efficiency and approximation

A simple example

Consider:

$$f(x) = 5x_1x_2 + 3x_1x_4 + 7x_3x_4 \text{ over } S := [0,1]^4$$

- By edge-concavity of $f(x)$, we have that concave envelope can be obtained by just examining the 2^4 extreme points.
- What if I add the term-wise concave envelopes?

$$\begin{aligned} g(x) &= \{5w_1 + 3w_2 + 7w_3 \,| \\ & \quad w_1 = \text{conv}_{[0,1]^2}(x_1x_2)(x), \\ & \quad w_2 = \text{conv}_{[0,1]^2}(x_1x_4)(x), \\ & \quad w_3 = \text{conv}_{[0,1]^2}(x_3x_4)(x)\} \end{aligned}$$

How good of an approximation is $g(x)$ of $\text{conv}_{[0,1]^4}(f)(x)$?

“Positive” result about “positive” coefficients

Theorem [Crama (1993)], [Coppersmith, Günlük, Lee, Leung (1999)], [Meyer, Floudas (2005)]

Consider the function $f(x) : [0,1]^n \to \mathbb{R}$ given by:

$$f(x) = \sum_{(i,j)\in E} a_{ij} x_i x_j$$

If $a_{ij} \geq 0 \ \forall (i,j) \in E$, then the concave envelope of f is given by (weighted) sum of the concave envelope of the individual functions $x_i x_j$.

Proof: Thanks total unimodularity!

$$f(x) = 5x_1x_2 + 3x_1x_4 + 7x_3x_4 \text{ over } S := [0,1]^4$$

$$\begin{aligned} g(x) \quad = \quad \max \quad & 5w_1 + 3w_2 + 3w_3 \\ \text{s.t.} \quad & w_1 \le x_1, w_1 \le x_2 \\ & w_2 \le x_1, w_2 \le x_4 \\ & w_3 \le x_3, w_3 \le x_4 \\ & 1 \ge w \ge 0. \end{aligned}$$

- Lets say we are computing concave envelope at $\hat{x}$ of f. Let $\hat{w}$ be the optimal solution of the above.
- g is concave function: $g(\hat{x}) \ge \text{conc}_{[0,1]^4} f(x)(\hat{x})$.
- By TU matrix treating x, w as variables (and therefore integrality of the polytope in the x, w space), $(\hat{x}, \hat{w}) = \sum_k \lambda_k (x^k, w^k)$ where (x^k, w^k) are integral and $\lambda \in \Delta$.
- $g(\hat{x}) = 5\hat{w}_1 + 3\hat{w}_2 + 7\hat{w}_3 = \sum_k \lambda_k (5w_1^k + 3w_2^k + 7w_3^k) \le \text{conc}_{[0,1]^4} f(x)(\hat{x})$.

More generally...

- Given $f(x) = \sum_{(i,j)\in E} a_{ij}x_i x_j$ and a particular $\hat{x} \in [0,1]^n$ let:

$$\text{ideal}(\hat{x}) = \text{conc}_{[0,1]^n}(f)(\hat{x}) - \text{conv}_{[0,1]^n}(f)(\hat{x})$$

and

$$\text{efficient}(\hat{x}) = \text{McCormick Upper}(f)(\hat{x}) - \text{McCormick Lower}(f)(\hat{x})$$

- Clearly $\text{efficient}(\hat{x}) \geq \text{ideal}(\hat{x})$.

How much larger (worse) is efficient($\hat{x}$) in comparison to ideal($\hat{x}$)?

Answers

- Consider the graph $G(V, E)$ where V is the set of nodes and E is the set of terms $x_i x_j$ in the function f for which $a_{ij} \neq 0$.
- Let the weight of edge (i, j) be a_{ij}.

Theorem

ideal$(\hat{x})$ = efficient$(\hat{x})$ *for all* $\hat{x} \in [0,1]^n$ *iff* G *is bipartite and each cycle have even number of positive weights and even number of negative weights.*

- [Luedtke, Namazifar, Linderoth (2012)]
- [Misener, Smadbeck, Floudas (2014)]
- [Boland, D., Kalinowski, Molinaro, Rigterink (2017)]

More Answers...

Theorem ([Luedtke, Namazifar, Linderoth (2012)])

If $a_{ij} \geq 0$, *then*

$$\text{ideal}(\hat{x}) \leq \text{efficient}(\hat{x}) \leq \left(2 - \frac{1}{\lceil \chi(G)/2 \rceil}\right) \cdot \text{ideal}(\hat{x}),$$

where $\chi(G)$ *is the chromatic number of the graph (minimum number of colors needed to color the vertices, so that no two vertices connected by an edge have the same color).*

Theorem ([Boland, D., Kalinowski, Molinaro, Rigterink (2017)])

In general,

$$\text{ideal}(\hat{x}) \leq \text{efficient}(\hat{x}) \leq 600\sqrt{n} \cdot \text{ideal}(\hat{x}),$$

where the multipicative ratio is tight upto constants.

6.1
Proofs for the case $a_{ij} \geq 0$

Infinite to finite

Theorem ([Luedtke, Namazifar, Linderoth (2012)])

If $a_{ij} \geq 0$, then

$$\text{ideal}(\hat{x}) \leq \text{efficient}(\hat{x}) \leq \left(2 - \frac{1}{\lceil \chi(G)/2 \rceil}\right) \cdot \text{ideal}(\hat{x}),$$

where $\chi(G)$ is the chromatic number of the graph (minimum number of colors needed to color the vertices, so that no two vertices connected by an edge have the same color).

(Non-trivial) part of Theorem is equivalent to:

$$\min_{\hat{x} \in [0,1]^n} \left(\left(2 - \frac{1}{\lceil \chi(G)/2 \rceil}\right) \cdot \text{ideal}(\hat{x}) - \text{efficient}(\hat{x}) \right) \geq 0$$

Step 1: Infinite to finite

$$\min_{\hat{x} \in [0,1]^n} \left(\left(2 - \frac{1}{\lceil \chi(G)/2 \rceil} \right) \cdot \text{ideal}(\hat{x}) - \text{efficient}(\hat{x}) \right) \geq 0$$

First task:
It is sufficient to prove:

$$\min_{\hat{x} \in \{0, \frac{1}{2}, 1\}^n} \left(\left(2 - \frac{1}{\lceil \chi(G)/2 \rceil} \right) \cdot \text{ideal}(\hat{x}) - \text{efficient}(\hat{x}) \right) \geq 0$$

Let $\boxed{\rho := \left(2 - \frac{1}{\lceil \chi(G)/2 \rceil} \right) \geq 1}$

Step 1: Infinite to finite

$$\begin{aligned} & \min_{\hat{x} \in [0,1]^n} && (\rho \cdot \text{ideal}(\hat{x}) - \text{efficient}(\hat{x})) \\ = & \min_{\hat{x} \in [0,1]^n} && (\rho \cdot \text{conc}_{[0,1]^n}(f)(\hat{x}) - \rho \cdot \text{conv}_{[0,1]^n}(f)(\hat{x}) \\ & && - \text{McCormick Upper}(f)(\hat{x}) + \text{McCormick Lower}(f)(\hat{x})) \end{aligned}$$

However, since $a_{ij} \geq 0$, we have already seen:

$\boxed{\text{conc}_{[0,1]^n}(f)(\hat{x}) = \text{McCormick Upper}(f)(\hat{x})}$, so:

$$\begin{aligned} = & \min_{\hat{x} \in [0,1]^n} && ((\rho - 1) \cdot \text{conc}_{[0,1]^n}(f)(\hat{x}) - \rho \cdot \text{conv}_{[0,1]^n}(f)(\hat{x}) \\ & && + \text{McCormick Lower}(f)(\hat{x})) \end{aligned}$$

Step 1: Infinite to finite

Let

$$\text{MC} := \left\{ (x,y) \in [0,1]^n \times [0,1]^{n(n-1)/2} \;\middle|\; \begin{array}{lll} y_{ij} & \geq & 0, \\ y_{ij} & \geq & x_i + x_j - 1, \\ y_{ij} & \leq & x_i, \\ y_j & \leq & x_j \end{array} \quad \forall i,j \in [n] (i \neq j) \right\}$$

$$\begin{array}{rll} = & \min_{\hat{x} \in [0,1]^n} & ((\rho - 1) \cdot \text{conc}_{[0,1]^n}(f)(\hat{x}) - \rho \cdot \text{conv}_{[0,1]^n}(f)(\hat{x}) \\ & & + \text{McCormick Lower}(f)(\hat{x})) \\ = & \min_{(\hat{x},\hat{y}) \in MC} & ((\rho - 1) \cdot \text{conc}_{[0,1]^n}(f)(\hat{x}) - \rho \cdot \text{conv}_{[0,1]^n}(f)(\hat{x}) \\ & & + \sum_{(i,j) \in E} a_{ij} y_{ij}) \end{array}$$

- $\rho - 1 \geq 0$ implies, $(\rho - 1) \cdot \text{conc}_{[0,1]^n}(f)$ is concave.
- $\text{conv}_{[0,1]^n}(f)$ is convex, so $-\rho \cdot \text{conv}_{[0,1]^n}(f)$

So the optimal solution can be assumed to be at a vertex of MC!

Step 1: Infinite to finite

Let

$$\text{MC} := \left\{ (x,y) \in [0,1]^n \times [0,1]^{n(n-1)/2} \;\middle|\; \begin{array}{lll} y_{ij} & \geq & 0, \\ y_{ij} & \geq & x_i + x_j - 1, \\ y_{ij} & \leq & x_i, \\ y_j & \leq & x_j \end{array} \quad \forall i,j \in [n](i \neq j) \right\}$$

Proposition [Padberg (1989)]

All the extreme points of MC are in $\{0, \frac{1}{2}, 1\}^n$

So:

$$\begin{array}{ll} & \min_{\hat{x} \in [0,1]^n} \left(\left(2 - \frac{1}{\lceil \chi(G)/2 \rceil}\right) \cdot \text{ideal}(\hat{x}) - \text{efficient}(\hat{x}) \right) \geq 0 \\ \Leftrightarrow & \min_{\hat{x} \in \{0, \frac{1}{2}, 1\}^n} \left(\left(2 - \frac{1}{\lceil \chi(G)/2 \rceil}\right) \cdot \text{ideal}(\hat{x}) - \text{efficient}(\hat{x}) \right) \geq 0 \end{array}$$

Step 2: Computation of efficient($\hat{x}$)

Notation:

- Remember $G(V, E)$
- For U^1, U^2, $\delta(U^1, U^2)$ is the edges of G where one end point is in U^1 and the other end point in U^2.
- Corresponding to $\hat{x} \in \left\{0, \frac{1}{2}, 1\right\}$, let $V := V_0 \cup V_f \cup V_1$

Proposition

For $\hat{x} \in \left\{0, \frac{1}{2}, 1\right\}$, efficient$(\hat{x}) = \frac{1}{2} \sum_{(i,j) \in \delta(V_f, V_f)} a_{ij}$.

- This is just calculation, remembering that the MC concave and convex envelope 'cancel out for y_{ij} if x_i or x_j are in $\{0, 1\}$'.

Step 3: Estimation of ideal($\hat{x}$): $\text{conc}_{[0,1]^n}(f)(\hat{x})$

$$\text{ideal}(\hat{x}) = \text{conc}_{[0,1]^n}(f)(\hat{x}) - \text{conv}_{[0,1]^n}(f)(\hat{x})$$

First estimate $\text{conc}_{[0,1]^n}(f)(\hat{x})$:

Proposition

For $\hat{x} \in \{0, \frac{1}{2}, 1\}$, $\text{conc}_{[0,1]^n}(f)(\hat{x}) =$
$\sum_{(i,j)\in\delta(V_1,V_1)} a_{ij} + \frac{1}{2}\sum_{(i,j)\in\delta(V_1,V_f)} a_{ij} + \frac{1}{2}\sum_{(i,j)\in\delta(V_f,V_f)} a_{ij}$.

Step 3: Estimation of ideal($\hat{x}$): $\text{conv}_{[0,1]^n}(f)(\hat{x})$

Now we want to estimate $\text{conv}_{[0,1]^n}(f)(\hat{x})$

- Remember $G(V,E)$ and $V := V_1 \cup V_f \cup V_0$.
- Suppose $T_f^a \cup T_f^b$ is a partition of the nodes in T_f. Then:
- Note $\hat{x} = \frac{1}{2} \cdot x(T_1 \cup T_f^a) + \frac{1}{2} \cdot x(T_1 \cup T_f^b)$
- Therefore $\text{conv}_{[0,1]^n}(f)(\hat{x}) \leq$ $\frac{1}{2}\text{conv}_{[0,1]^n}(f)(x(T_1 \cup T_f^a)) + \frac{1}{2}\text{conv}_{[0,1]^n}(f)(x(T_1 \cup T_f^a))$.
- With some simple calculations:

$$\frac{1}{2}\text{conv}_{[0,1]^n}(f)(x(T_1 \cup T_f^a)) + \frac{1}{2}\text{conv}_{[0,1]^n}(f)(x(T_1 \cup T_f^a) = \frac{1}{2}(A + B + C - D),$$

where:

- $A = 2\sum_{(i,j)\in\delta(T_1,T_1)} a_{ij}$
- $B = \sum_{(i,j)\in\delta(T_1,T_f)} a_{ij}$
- $C = \sum_{(i,j)\in\delta(T_f,T_f)} a_{ij}$
- $D = \sum_{(i,j)\in\delta(T_f^a,T_b^b)} a_{ij}$ < − − − This is a cut among the fractional vertices! Question: how large can this cut be?

Step 3: Estimation of ideal($\hat{x}$): $\text{conv}_{[0,1]^n}(f)(\hat{x})$

Theorem

Assuming $a_{ij} \geq 0$ for all $(i,j) \in E$, there exists a cut of value at least:

$$\frac{1}{2}\left(\frac{1}{2} + \frac{1}{2\chi(G) - 2}\right) \sum_{(i,j)\in E} a_{ij}$$

- Apply this Theorem to the induced subgraph of fractional vertices.
- Note that the chromatic number cannot increase for a subgraph.

Putting it all together

- Examining $\hat{x} \in \left\{0, \frac{1}{2}, 1\right\}$:
- efficient$(\hat{x}) = \frac{1}{2}\sum_{(i,j)\in\delta(V_f,V_f)} a_{ij}$.
-

$$\begin{aligned}\text{ideal}(\hat{x}) \quad \geq \quad & \sum_{(i,j)\in\delta(V_1,V_1)} a_{ij} + \frac{1}{2}\sum_{(i,j)\in\delta(V_1,V_f)} a_{ij} \\ & + \frac{1}{2}\sum_{(i,j)\in\delta(V_f,V_f)} a_{ij} \\ & - \sum_{(i,j)\in\delta(V_1,V_1)} a_{ij} - \frac{1}{2}\sum_{(i,j)\in\delta(V_1,V_f)} a_{ij} \\ & - \frac{1}{4}\sum_{(i,j)\in\delta(V_f,V_f)} a_{ij} \\ & + \frac{1}{4\chi(G)-4}\sum_{(i,j)\in\delta(V_f,V_f)} a_{ij}\end{aligned}$$

- ideal$(\hat{x}) \geq \frac{1}{4}\left(1 + \frac{1}{\chi(G)-1}\right) \cdot \sum_{(i,j)\in\delta(V_f,V_f)} a_{ij}$.
- $\frac{\text{efficient}(\hat{x})}{\text{ideal}(\hat{x})} \leq \frac{2\chi(G)-2}{\chi(G)}$.

Mixed a_{ij} case

Theorem ([Boland, D., Kalinowski, Molinaro, Rigterink (2017)])

In general,

$$\text{ideal}(\hat{x}) \leq \text{efficient}(\hat{x}) \leq 600\sqrt{n} \cdot \text{ideal}(\hat{x}),$$

where the multipicative ratio is tight upto constants.

Similar techniques, a key result on cuts of graphs:

Theorem ([Boland, D., Kalinowski, Molinaro, Rigterink (2017)])

Let $G = (V, E)$ be a complete graph on vertices $V = \{1, \dots, n\}$ and let $a \in \mathbb{R}^{n(n-1)/2}$ be edge weights. Then ther exists a $U \subseteq V$ such that

$$\left| \sum_{(i,j) \in \delta(U, V \setminus U)} a_{ij} \right| \geq \frac{1}{600\sqrt{n}} \cdot \sum_{(i,j) \in E} |a_{ij}|$$

Thank You!